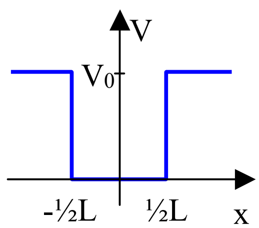
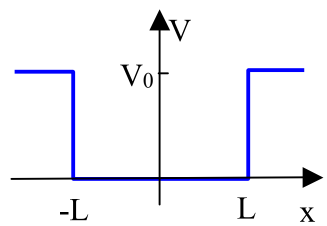
1. The diagram below shows two finite square wells. How would you expect the energy levels in (b) compare to the levels in (a)?

(a)



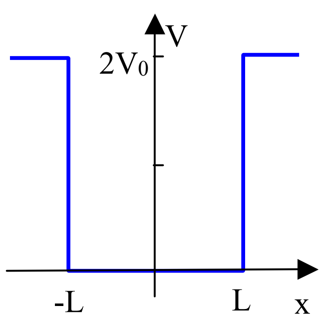
(b)



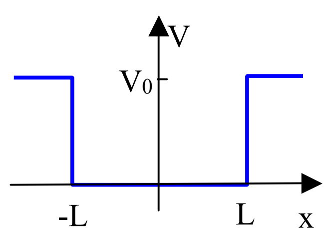
The wavelength of the lowest energy level approximately the width of the well, so the wavelength in (b) should be approximately double that in (a) for the same eigenstate. As a result, the wave number halves for each level. Since $E = \frac{\hbar^2 k^2}{2m}$, thus $E_b \cong \frac{1}{4}E_a$.

2. Consider the two finite square wells below. How would you expect the energy levels in (b) compare to the levels in (a)? Which will be larger?

(a)

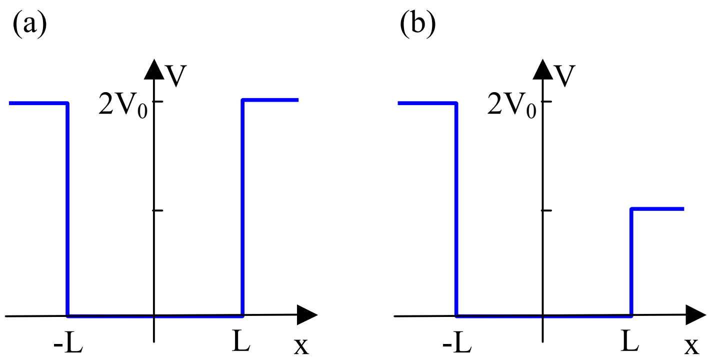


(b)



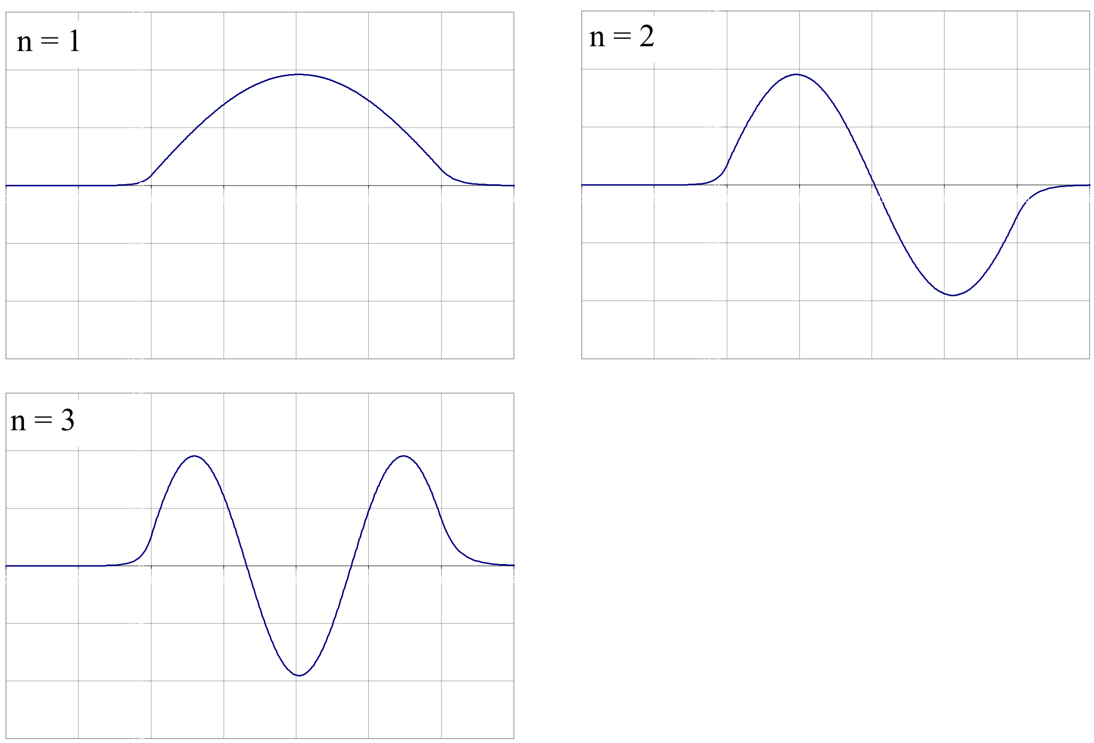
The wavelength of the lowest energy level approximately the width of the well, so the wavelength in (b) should be approximately equal to that in (a) for the same eigenstate. There is a difference in the behaviour of $\psi(x)$ in the region outside the well. For (a), $\psi(x)$ can be expected to decay much faster than in (b) since V – E is much greater. Thus the wavelength in (b) will be slight longer than in (a). As a result, the wave number in (b) is slightly smaller than in (a). Since $E = \frac{\hbar^2 k^2}{2m}$, thus $E_b \le E_a$.

3. Consider the two finite square wells below. How would you expect the energy levels in (b) compare to the levels in (a)? Which will be larger? Sketch ψ(x) for the first three levels in (b).

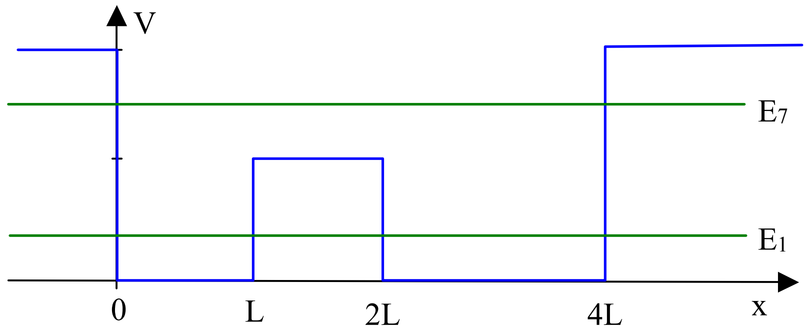


The wavelength of the lowest energy level approximately the width of the well, so the wavelength in (b) should be approximately equal to that in (a) for the same eigenstate. There is a difference in the behaviour of ψ(x) in the region outside the well. For (a), ψ(x) can be expected to decay much faster than in (b) on the right-hand side since V – E is much greater. Thus the wavelength in (b) will be slight longer than in (a). As a result, the wave number in (b) is slightly smaller than in (a). Since $E = \frac{\hbar^2 k^2}{2m}$, thus $E_b <= E_a$.

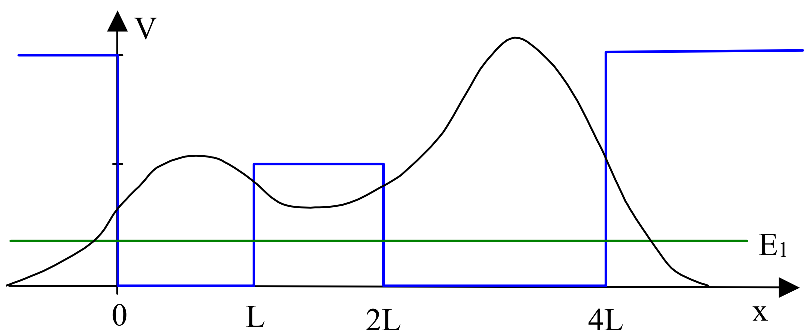
The first three energy levels would look like



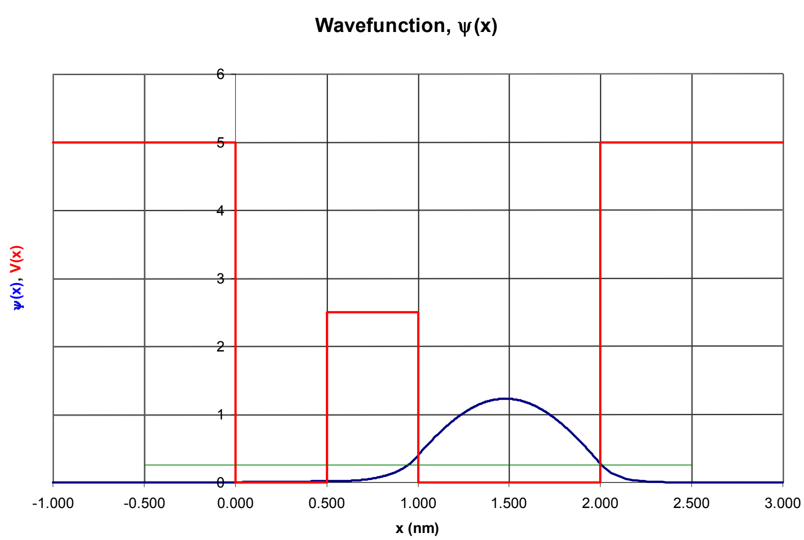
4. Sketch $\psi(x)$ for the following potential well at the indicated energy levels.



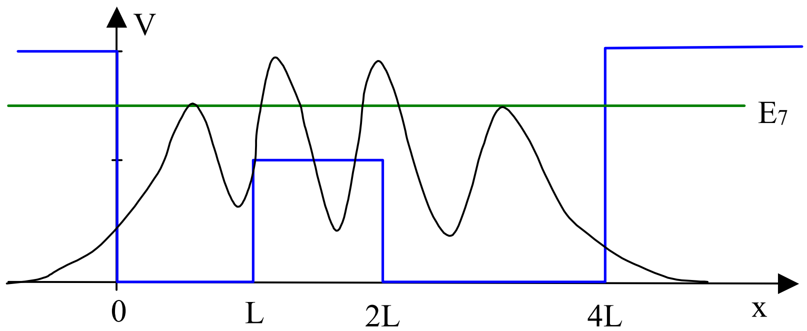
For E_1, in the regions $x < 0$ and $x > 0$ the exponential decay will be at the same rate. For $x > L$ and $x < 2L$, $V > E_1$ so we get exponential terms but they must meet. For the sinusoidal regions, $0 < x < L$ and $2L < x < 4L$, at energy E_1, we have a double well. Therefore we expect one antinode in each well. Now λ is approximately L in the left well and $\lambda \cong 2L$. Since kinetic energy is proportional to $1/\lambda^2$, E_K is bigger on the left than on the right, so the amplitude is higher on the left. We expect something like



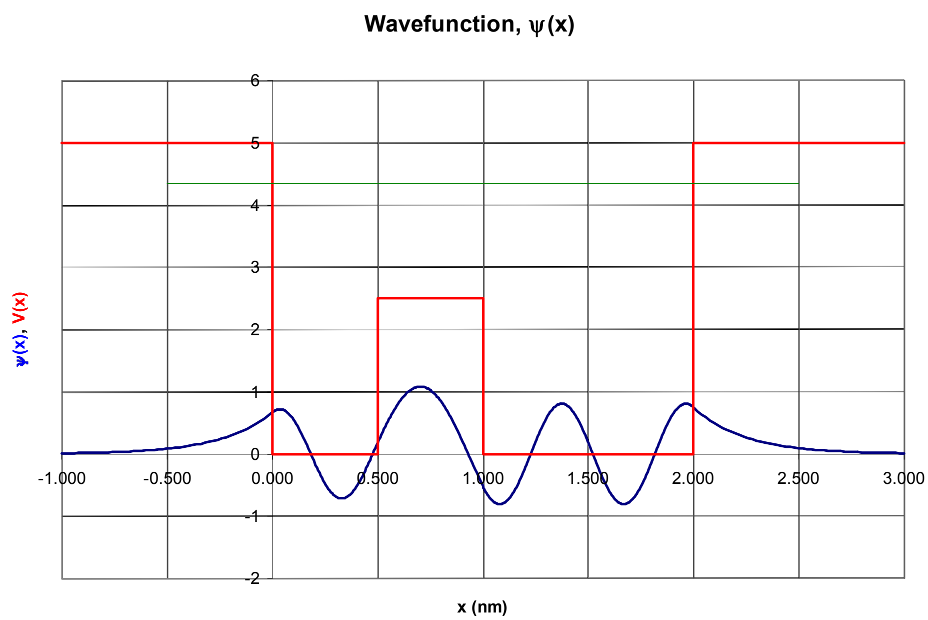
Solving an actual well, we get the following diagram. Note that there is hardly any noticeable bump in the first region.



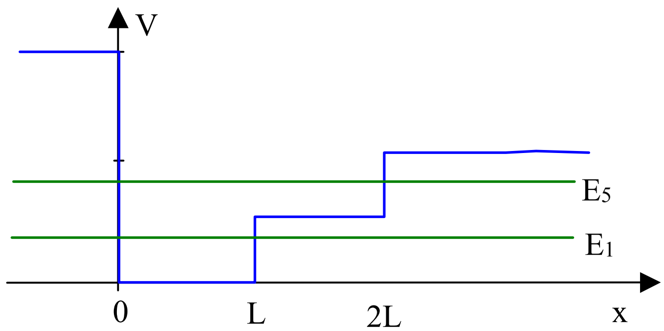
For E_7, the entire region $0 < x < 4L$ is sinusoidal. We will need seven antinodes. The wavelength with be large is the regions $0 < x < L$ and $2L < x < 4L$ but small in the region $L < x < 2L$. Also the amplitude will be larger in $L < x < 2L$ than elsewhere. We expect something like



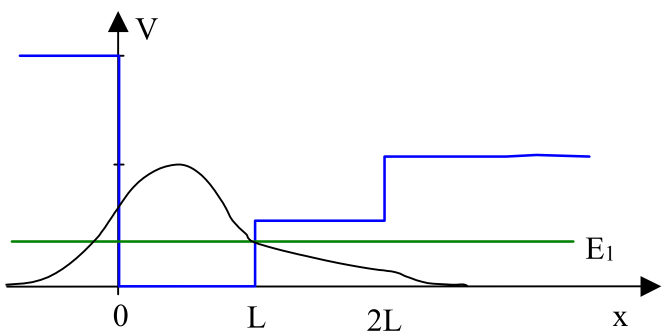
A exact solution to a similar well is shown below and has the general feature we have discussed.



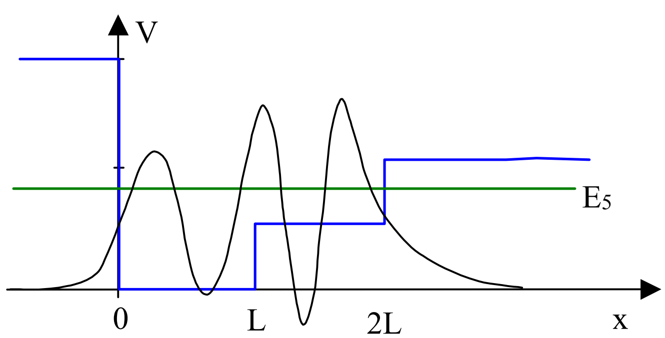
5. Sketch $\psi(x)$ for the following potential well at the indicated energy levels.



For the first energy level E_1, the exponential decay regions are somewhat different. To the right it will take longer to decay. In the sinusoidal region, we have an ordinary well, so we expect one simple antinode.

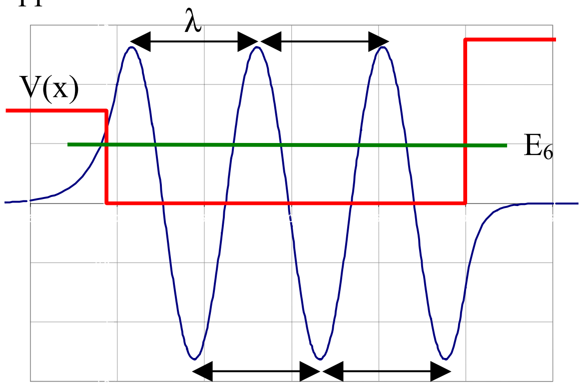


For the fifth energy level E_5, the exponential decay regions are the different with the decay faster on the left. In the sinusoidal region, we expect five antinodes. For the region $L < x < 2L$, KE is smaller so the wavelength is larger as is the amplitude as compared to the region $0 < x < L$. Thus we expect

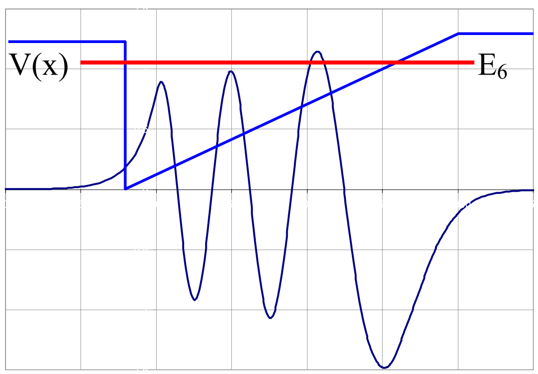


6. Sketch the V(x) for the following ψ(x). Identify n, the energy level.

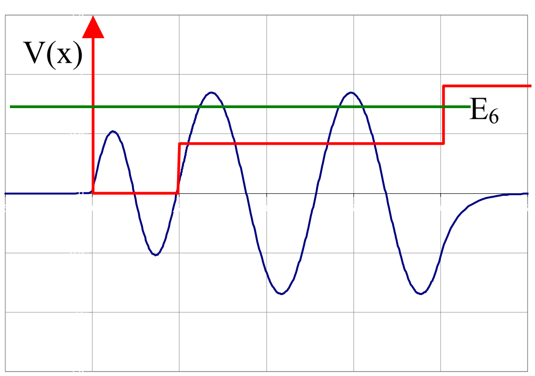
(a) There are six antinodes so this is E_6. The amplitude and wavelength doesn't appear to be changing so this is a flat-bottomed well. The right hand side is decaying faster, so we appear to have



(b) We have six antinodes so this is E_6. The amplitude increases from left to right and the wavelength decreases. The well has a ramp. The sides seem to be decaying at about the same rate.



(c) There are six antinodes so this is E_6. The left side is discontinuous, the sign of an infinite potential on that side. The amplitude is high and wavelength is small for the first two nodes suggesting high KE then amplitude decreases and wavelength increases indicating low KE. The potential well appears to have two levels. Therefore it probably looks like



7. For $\psi_3(x)$ in the infinite square well potential, determine $\langle x\rangle$, $\langle x^2\rangle$, and σ_x.

For the infinite square well, the third level is given by

$$\psi_3(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right) & 0 \le x \le L \\ 0 & otherwise \end{cases}$$

The expectation value of x is

$$\langle x\rangle = \bar{x} = \int_0^L \psi_3^*(x)x\psi_3(x)dx = \frac{L}{2}.$$

The expectation value of x^2 is

$$\langle x^2\rangle = \overline{x^2} = \int_0^L \psi_3^*(x)x^2\psi_3(x)dx = \left(\frac{1}{3} - \frac{1}{18\pi^2}\right)L^2.$$

The uncertainty in x is given by

$$\sigma_x = \sqrt{\overline{x^2} - \bar{x}^2} = \sqrt{3\pi^2 - 2}\,\frac{L}{6\pi} \cong 0.27876\,L.$$

8. For $\psi_3(x)$ in the infinite square well potential, determine <p>, <p^2>, and σ_p.

First we have to see what p_{op} and $(p_{op})^2$ do to $\psi_3(x)$. We have

$$p_{op}\psi_3(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right) = -i\hbar\frac{3\pi}{L}\sqrt{\frac{2}{L}}\cos\left(\frac{3\pi x}{L}\right).$$

Next

$$(p_{op})^2\psi_3(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{3\pi}{L}\sqrt{\frac{2}{L}}\cos\left(\frac{3\pi x}{L}\right)\right) = \hbar^2\left(\frac{3\pi}{L}\right)^2\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi x}{L}\right).$$

So the expectation value of p is

$$\langle p\rangle = \overline{p} = \int_0^L \psi_3^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_3(x)\,dx = 0.$$

Similarly the expectation value of p^2 is

$$\langle p^2\rangle = \overline{p^2} = \int_0^L \psi_3^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi_3(x)\,dx = \frac{9\pi^2\hbar^2}{L^2}.$$

The uncertainty in p is given by

$$\sigma_p = \sqrt{\overline{p^2} - \overline{p}^2} = \frac{3\pi\hbar}{L}.$$

Using the value of σ_x from the previous question, we can check the value of $\sigma_x\sigma_p$. We find

$$\sigma_x\sigma_p = \Delta x\Delta p = \sqrt{\frac{3}{4}\pi^2 - \frac{1}{2}}\hbar \cong 2.63\hbar > \frac{\hbar}{2}.$$

9. For $\psi_3(x)$ in the finite square well potential, determine $\langle x \rangle$, $\langle x^2 \rangle$, and σ_x. Use MAPLE if you wish. Take $V = 8$ eV, $L = 1$ nm, $E_3 = 2.573490$ eV, and $m_e = 0.511$ MeV.

As we have seen the even wavefunction solutions to a finite square well are given by

$$\psi_3(x) = \begin{cases} Ae^{\alpha x} & x < -\dfrac{L}{2} \\ B\cos(kx) & -\dfrac{L}{2} \le x \le \dfrac{L}{2} \\ Ae^{-\alpha x} & x > \dfrac{L}{2} \end{cases}$$

Where $k = \sqrt{\frac{2m}{\hbar^2}E}$ and $\alpha = \sqrt{\frac{2m}{\hbar^2}(V-E)}$. Continuity at the boundary indicates that

$A = Be^{-\alpha L/2}\cos\left(\frac{kL}{2}\right)$. The value of the constant B is set by the normalization requirement

that $\int_{-\infty}^{\infty}\psi_3^*(x)\psi_3(x)\,dx = 1$. Evaluation of the integral yields $B = 0.764063$.

The expectation value of x is

$$\langle x \rangle = \bar{x} = \int_{-\infty}^{\infty}\psi_3^*(x)x\psi_3(x)dx = 0.$$

The expectation value of x^2 is

$$\langle x^2 \rangle = \overline{x^2} = \int_{-\infty}^{\infty}\psi_3^*(x)x^2\psi_3(x)dx = 0.0362522\,nm^2$$

The uncertainty in x is given by

$$\sigma_x = \sqrt{\overline{x^2} - \bar{x}^2} \cong 0.190400\,nm.$$

10. For $\psi_3(x)$ in the finite square well potential, determine $\langle p \rangle$, $\langle p^2 \rangle$, and σ_p. Use MAPLE if you wish. Take $V = 8$ eV, $L = 1$ nm, $E_3 = 2.573490$ eV, and $m_e = 0.511$ MeV.

First we have to see what p_{op} and $(p_{op})^2$ do to $\psi_3(x)$ shown in the previous question. We know $p_{op}\psi_3(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_3(x)$. Thus

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_3(x) = \begin{cases} \frac{\hbar}{i}\alpha A e^{\alpha x} & x < -\frac{L}{2} \\ -\frac{\hbar}{i}kB\sin(kx) & -\frac{L}{2} \le x \le \frac{L}{2} \\ -\frac{\hbar}{i}\alpha A e^{-\alpha x} & x > \frac{L}{2} \end{cases}$$

Also $(p_{op})^2\psi_3(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi_3(x)$. Hence

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi_3(x) = \begin{cases} -\hbar^2\alpha^2 A e^{\alpha x} & x < -\frac{L}{2} \\ \hbar^2 k^2 B\cos(kx) & -\frac{L}{2} \le x \le \frac{L}{2} \\ -\hbar^2\alpha^2 A e^{-\alpha x} & x > \frac{L}{2} \end{cases}$$

$$\langle p \rangle = \overline{p} = \int_{-\infty}^{\infty}\psi_3^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_3(x)\,dx = 0.$$

Similarly the expectation value of p^2 is

$$\langle p^2 \rangle = \overline{p^2} = \int_{-\infty}^{\infty}\psi_3^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi_3(x)\,dx \cong 19.716399\hbar^2.$$

The uncertainty in p is given by

$$\sigma_p = \sqrt{\overline{p^2} - \overline{p}^2} \cong 4.440315\hbar.$$

Using the value of σ_x from the previous question, we can check the value of $\sigma_x\sigma_p$. We find

$$\sigma_x\sigma_p = \Delta x\Delta p \cong 0.845436\hbar > \frac{\hbar}{2}.$$

11. For $\psi_2(x)$ in the simple harmonic oscillator potential, determine $\langle x\rangle$, $\langle x^2\rangle$, and σ_x. Use MAPLE if you wish.

The text gives $\psi_2(x)$ as $\psi_2(x) = A_2\left(1-\frac{2m\omega x^2}{\hbar}\right)e^{-m\omega x^2/2\hbar}$. We need to normalize the function to find A_2. Evaluating $\int_{-\infty}^{\infty}\psi_2^*(x)\psi_2(x)\,dx = 1$ yields $A_2 = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}$.

The expectation value of x is

$$\langle x\rangle = \bar{x} = \int_{-\infty}^{\infty}\psi_2^*(x)x\psi_2(x)dx = 0.$$

The expectation value of x^2 is

$$\langle x^2\rangle = \overline{x^2} = \int_{-\infty}^{\infty}\psi_2^*(x)x^2\psi_2(x)dx = \frac{5\hbar}{2m\omega}.$$

The uncertainty in x is given by

$$\sigma_x = \sqrt{\overline{x^2} - \bar{x}^2} = \sqrt{\frac{5\hbar}{2m\omega}}.$$

12. For $\psi_2(x)$ in the simple harmonic oscillator potential, determine $\langle p \rangle$, $\langle p^2 \rangle$, and σ_p. Use MAPLE if you wish.

First we have to see what p_{op} and $(p_{op})^2$ do to $\psi_2(x)$. We have

$$p_{op}\psi_2(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left[\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(1-\frac{2m\omega x^2}{\hbar}\right)e^{-m\omega x^2/2\hbar}\right] = im\omega x\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(5+\frac{2m\omega x^2}{\hbar}\right)e^{-m\omega x^2/2\hbar}.$$

Next

$$p_{op}^2\psi_2(x) = \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\left[\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(1-\frac{2m\omega x^2}{\hbar}\right)e^{-m\omega x^2/2\hbar}\right]$$

which is so complicated it is best left to MAPLE.

The expectation value of p is

$$\langle p \rangle = \overline{p} = \int_{-\infty}^{\infty}\psi_2^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_2(x)\,dx = 0.$$

Similarly the expectation value of p^2 is

$$\langle p^2 \rangle = \overline{p^2} = \int_{-\infty}^{\infty}\psi_2^*(x)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\psi_2(x)\,dx = \frac{5m\omega\hbar}{2}.$$

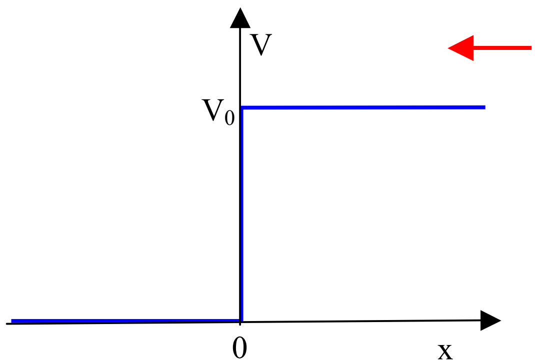
The uncertainty in p is given by

$$\sigma_p = \sqrt{\overline{p^2}-\overline{p}^2} = \sqrt{\frac{5m\omega\hbar}{2}}.$$

Using the value of σ_x from the previous question, we can check the value of $\sigma_x\sigma_p$. We find

$$\sigma_x\sigma_p = \Delta x\Delta p = \frac{5}{2}\hbar > \frac{\hbar}{2}.$$

13. Consider the step potential as shown below. Imagine that a beam of particles comes from the right instead of from the left as considered. Find expressions for R and T in this case.



In the incoming region, 0 < x the expression for the wavefunction is

$$\psi_I(x) = Ae^{iax} + Be^{-iax}.$$

The first term represents the incident particles moving to the right. The second term represents reflected particles moving to the left. Note that it is the discontinuity in potential not the barrier itself that leads to reflected particles. Note $a^2 = 2m(E-V)/\hbar^2$

In the transmission region, x <0 , the wavefunction will have the form

$$\psi_{II}(x) = Ce^{ikx}.$$

There will be no particles moving to the right. Note $k^2 = 2mE/\hbar^2$.

Continuity at x = 0 requires C = A + B and kC = a(A – B). Solving for C and B in terms of A we find C = 2aA/(a+k) and B = A(a-k)/(a+k). The fraction of reflected particles is given by

$$R = \frac{|B|^2}{|A|^2} = \left(\frac{a-k}{a+k}\right)^2.$$

The rest of the particles are transmitted so

$$T = 1 - R = \frac{4ak}{(a+k)^2}.$$

The form of the equations is identical to that for particles travelling the other way.