1. A particle is created in an accelerator. A detector, 20.0 m away in a straight line, records it 7.41×10^{-8} seconds later.
   a. How fast is the particle moving?
   b. What is β?
   c. How much time passes in the frame of the particle?
   d. How far has the particle travelled in its own frame?

   a. We can taking the accelerator and detector as being at rest in frame S. The distance and time are both therefore refer to S. In S, the particle has speed

$$v = \frac{d}{t} = \frac{20.0\,m}{7.41\times10^{-8}\,s} = 2.6991\times10^{8}\,m/s$$

   b. The velocity as a fraction of the speed of light, β, is

$$\beta = \frac{2.6991\times10^{8}\,m/s}{2.99792458\times10^{8}\,m/s} = 0.900$$

   and $\gamma = 2.2975$.

For the next two parts we need to define our events. For convenience let the creation of the particle be (0,0) in both frames, i.e. $(x_1', ct_1') = (x_1, ct_1) = (0,0)$, where the primed coordinates refer to the reference frame S′ of the particle. One event we know is the detection of the particle which is measured in the frame of the accelerator, i.e. $(x_2,t_2) = (20.0\text{ m}, 7.41 \times 10^{-8}\text{ s})$ or $(x_2, ct_2) = (20\text{ m}, 22.215\text{ m})$. We use the Lorentz Transformation formulas $x' = \gamma(x - \beta ct)$ and $ct' = \gamma(ct - vx/c^2)$ to find the events in the S′.

Event		S	S′
A	Particle Created	(0, 0)	(0, 0)
B	Particle Detected	(20, 22.215)	(0, 9.669)

   c. Note that in S′ the particle is at rest at the origin and the detector is approaching it. So events A and B happen at the same place in S′ and are recorded by the same clock as can be read from the table. The proper time interval in S′ is $ct' = 9.669$ m or $t' = 3.225 \times 10^{-8}$ seconds.

   Since the creation of the particle and the arrival of the detector occurs at the same place in S′, we could use the results of time dilation if we note $\Delta t' = \Delta t_P$. Then

$$\Delta t_P = \frac{\Delta t}{\gamma} = \frac{7.41\times10^{-8}}{2.2975} = 3.225\times10^{-8}\,s$$

Note: from the view of an observer in one IFR, times intervals as measured by observers moving at constant velocity appear shorter. This is called time dilation.

d. Asking how far has the particle travelled in its own frame is the same as asking where the detector is, x_2', at $t_2' = 0$. Since we know $x_2 = 20$ m, we need a formula relating, x', x, and t'. That would be $x_2 = \gamma(x_2' + \beta c t_2')$. So we find

$$20\,m = \gamma(x_2' + 0.9003c(0)) = 2.2975x_2'$$

Hence $x_2' = 8.705$ m.

We could have found this result by noting that since the particle sees the detector traveling at 0.9003c and since it takes 3.225×10^{-8} seconds to reach the particle, then it must have traveled a distance

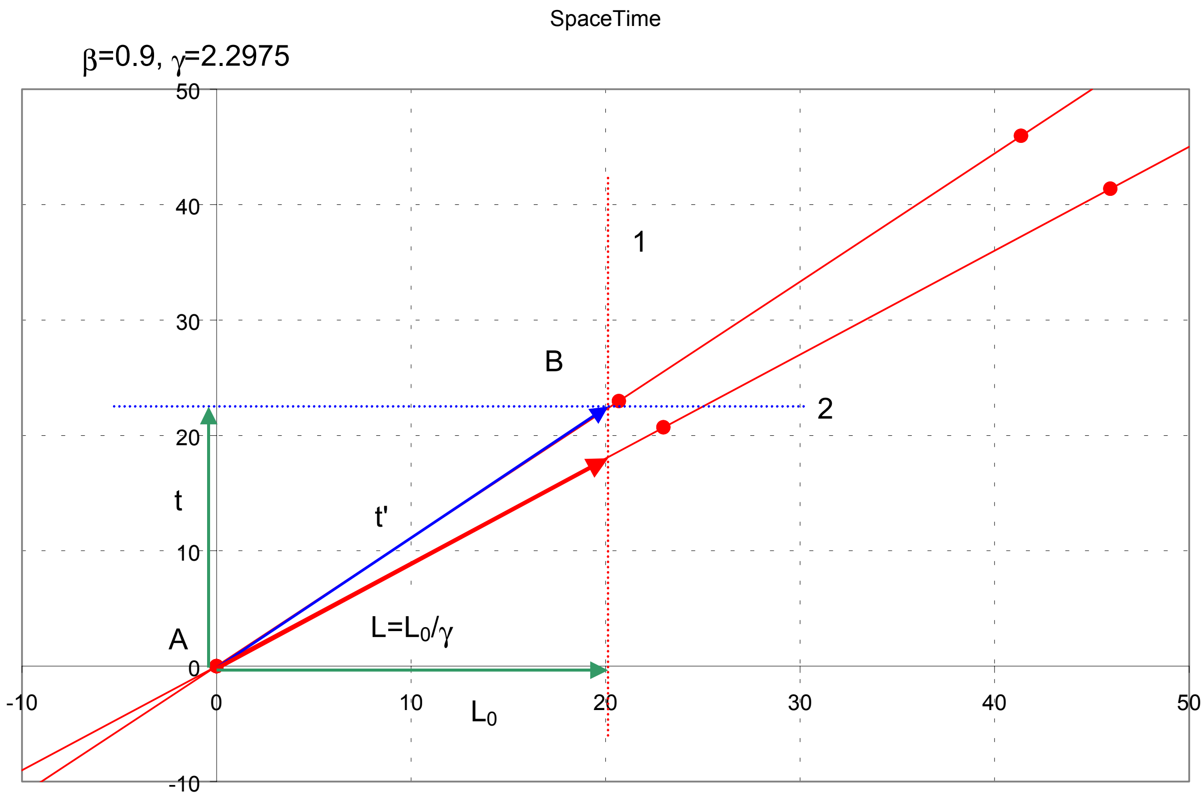
$$d' = 0.9003(2.997925 \times 10^{8}\text{ m/s})(3.225 \times 10^{-8}\text{ s}) = 8.705\text{ m.}$$

Or we could have noted that 20 m is the proper distance in S since the accelerator and detector are at rest, and use length contraction to find d'

$$L = \frac{L_P}{\gamma} = \frac{20\,m}{2.2975} = 8.705\,m$$

Note: from the view of an observer in one IFR, distances and lengths of objects moving at constant velocity appear shorter. This is called length contraction.

A spacetime sketch of the problem is shown below. The 10 m intervals in space and time in S′ are denoted by large dots on those axes. Note that 10 m in S′ is not the same as in S. Events A and B are labelled. The dashed red line labeled 1 indicates the location of the detector at all time. The distance between the accelerator and detector is a proper length. When line 1 intersects the x′ axis that gives the distance the particle sees measured, as required, at the same time ct′ = 0. The red arrow shows that distance. Note that it is about 8.7 m in S′. The intersection of the blue dashed line (t = 74.1 ns), line 2, with line 1 shows where the event occurs in both frames. Notice the intersection occurs on the ct′-axis at ct′ = 9.7 m and that that time is a proper time.


SpaceTime
β=0.9, γ=2.2975
50
40
30
20
10
0
-10
-10
0
10
20
30
40
50
1
2
B
A
t
t'
L=L0/γ
L0

2 You are timing a space ship race. Two spaceships both travel at 0.7c. As the first spaceship reaches the finish line, you start your stopwatch. The second ship arrives 1.5×10^{-5} seconds later.

a. From your frame of reference, how far behind was the losing spaceship?
b. From the losing spaceship's frame, how long after the winner did he cross the finish line?
c. From the winning spaceship's frame, how far behind was the loser?

It might seem that we are dealing with three frames of reference, but the two spaceships are traveling at the same speed in the same direction and so share the same IFR. They must agree with one another about time and distance measures. Also note $\gamma = 1.400$.

In your frame, the loser was simply

$$d = vt = (0.7c)(1.5 \times 10^{-5}\text{ s}) = 3.1478 \times 10^{3}\text{ m}$$

behind the winner. Since the rest of the questions involve transforming from one frame to another, we first set the first crossing the finish line as (0,0) in both frames, i.e. $(x_1', ct_1') = (x_1, ct_1) = (0,0)$. The other event that we know is the loser crossing the finish line $(x_2, t_2) = (0, 1.5 \times 10^{-5}\text{ s})$ or $(x_2, ct_2) = (0, 4497\text{ m})$.

Transforming all these events to the spaceships' frame yields

	Event	S	S′
A	Winner crosses line	(0, 0)	(0, 0)
B	When loser crosses	(0, 4497)	(–4407.8 m, 6296.9 m)

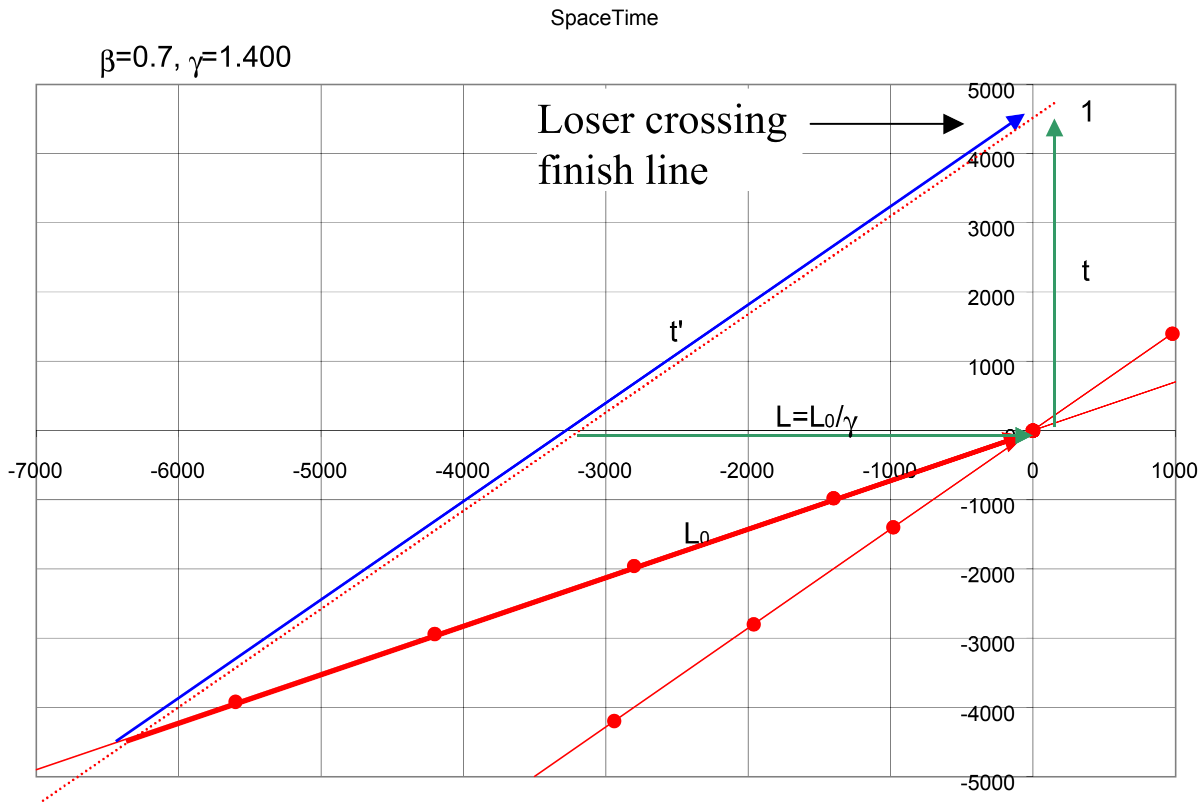
b. Notice in the table that events A and B occur at the same location (the winning spaceship being the origin) in S. So the time between these events is the proper time. Using time dilation, the spaceships will see a longer time interval given by

$$t' = \gamma t = (1.4003)(1.5 \times 10^{-5}\text{ s}) = 2.10 \times 10^{-5}\text{ s (or 6297 m)}.$$

c. The distance separating the two spaceships is just

$$d' = vt' = 0.7\, ct' = (0.7)(6297\text{ m}) = 4408\text{ m}.$$

The space-time diagram below shows the events. The large dots on the x' and ct' axes are 1000 m apart. The dashed line labeled 1 is parallel to the ct' axis and crosses the finish line at $x = 0$ (the ct axis). It therefore shows the path of the losing spaceship. Extrapolating back to the x'-axis ($ct' = 0$) we can see that it was about 4400 m behind when the winner crossed the finish line. An observer in S measuring that distance at the same time $ct = 0$, the origin and where line 1 intersects the x-axis, sees the smaller length given by the horizontal green arrow. The vertical green arrow is the proper time measured at $x = 0$ between the events. The long blue arrow shows the time perceived by the spaceship pilots.



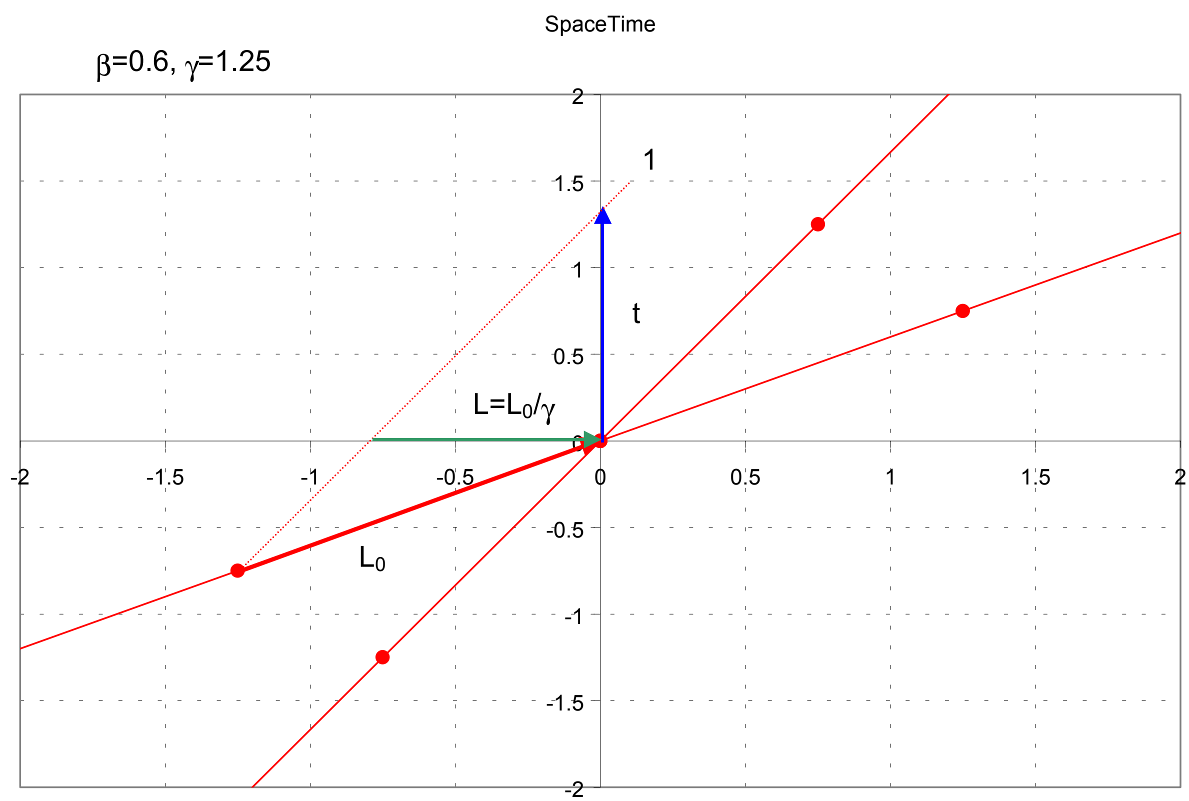
SpaceTime
β=0.7, γ=1.400
Loser crossing finish line
1
t
t'
L=L0/γ
L0
5000
4000
3000
2000
1000
-1000
-2000
-3000
-4000
-5000
-7000
-6000
-5000
-4000
-3000
-2000
-1000
0
1000

3. (Tipler, Modern Physics) A metrestick moves parallel with its length with speed v = 0.6 c relative to you. (a) Compute the length of the stick measured by you. (b) How long does it take for the stick to pass you? (c) Draw a spacetime-diagram from the viewpoint of your frame with the front of the meterstick at x = 0 when t = 0. Show how the aswers to (a) and (b) are obtained from the diagram.

(a) First $\beta = 0.6$ and $\gamma = 1.25$. In the frame where the metrestick is at rest, the metrestick is of course 1.00 m long (If it wasn't, an observer would know it was him that was moving). So 1.00 m is L_P the proper length of the stick. You, being stationary, see the moving object contracted by $L = L_P/\gamma = 1.00/1.25 = 0.80$ m.

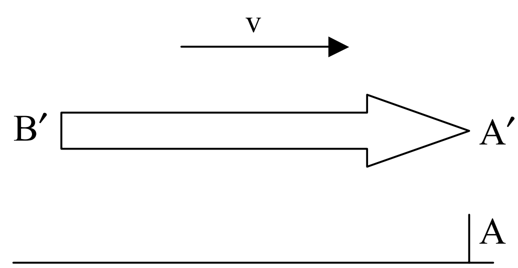
(b) You see the 0.80-metre stick travelling at v = 0.6 c, so it takes $t = L/v = 0.80/(0.6)(2.997925 \times 10^8) = 4.448 \times 10^{-9}$ s.

(c) The spacetime diagram is shown below. The tilted axes are the frame of the metrestick moving at $\beta = 0.6$. The large dots on the axes show $x' = \pm 1$ and $ct' = \pm 1$. The heavy red line on the x' axis is the metrestick which has length one metre. The nose is at the origin. The dashed line labelled 1 is where the tail of the metrestick is located at all time. We can only measure the metrestick in the stationary frame when both ends are measured at the same time. Since we know the nose is at the origin at x = 0 at t = 0, we look at where the dashed line 1 intersects the x-axis. The length is given by the green arrow and we see it is 0.8 m. To find out when the tail passes by we need to know when the tail is at x = 0. That occurs when the dashed line 1 intersects the ct-axis at about ct = 1.33 m or $t = 1.33/(2.997925 \times 10^8) = 4.45$ ns.



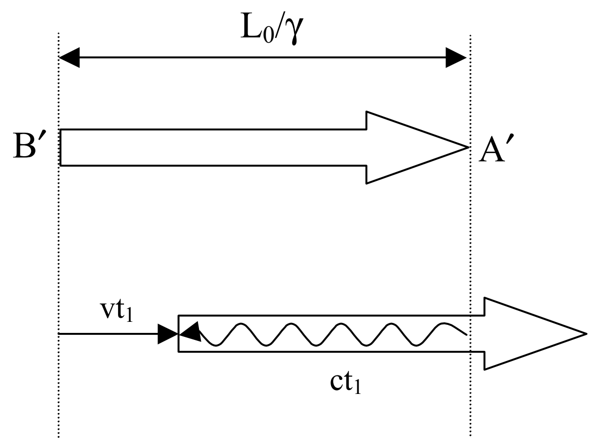
4. (From French, Special Relativity) A rocketship of proper length L_0 travels at constant velocity *v* relative to frame S (see figure – not included). The nose of the ship (A′) passes the point A in S at t= t′ = 0, and at this instant a light signal is sent from A′ to B′, the back of the spaceship.

(a) When, by rocketship time (t′), does the signal reach the tail (B′) of the ship?
(b) At what time t_1, as measured in S, does the signal reach the tail (B′) of the ship?
(c) At what time t_2, as measured in S, does the tail (B′) of the ship pass the point A?



(a) In S′ the spaceship is at rest. The distance the light has to travel is L_0. The light will take t′ = d/v = L_0/c. Note that t′ is not a proper time as the two events do not occur at the same place.

(b) In S, the back of the ship is moving forward to meet the oncoming light signal. Also the ship is smaller due to Length Contraction, $L_S = L_0/\gamma$. A sketch of the situation looks like the diagram below.



Clearly, $L_0/\gamma = vt_1 + ct_1$. Therefore the time is $t_1 = L_0/[\gamma(c+v)]$. (Note that if we had erroneously thought that the answer to part (a) was a proper time, we would have expected $t_1 = \gamma t' = \gamma L_0/c$.)

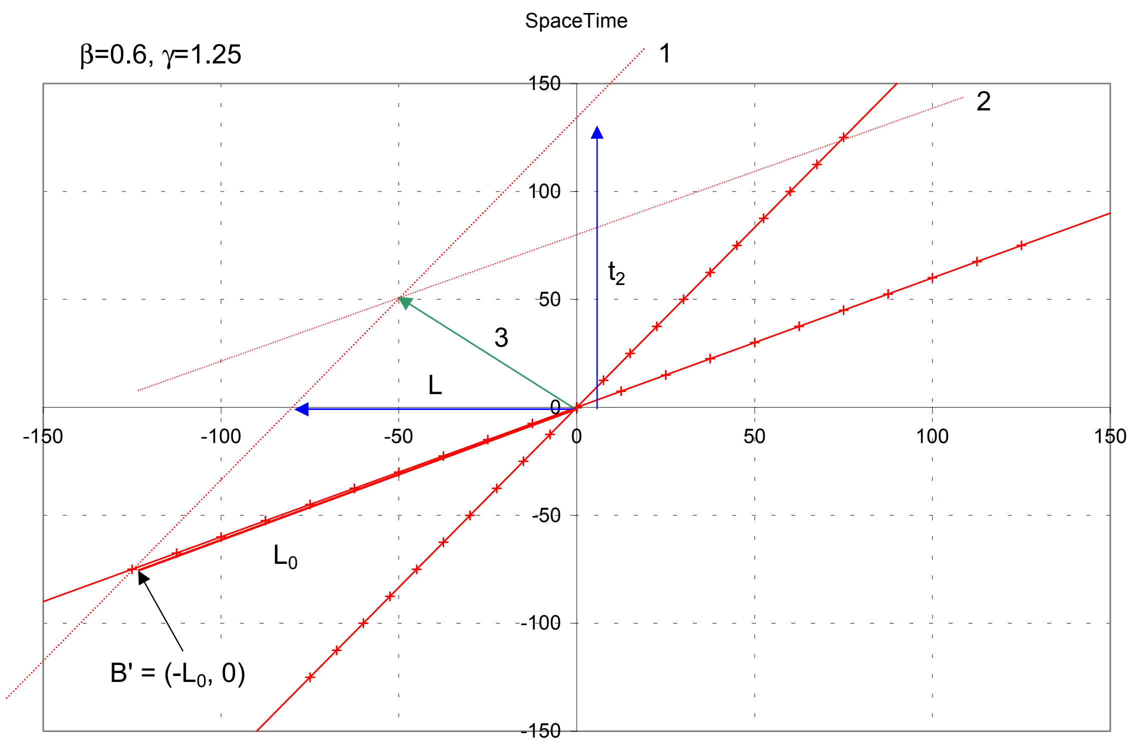
(d) In S, the tail of the spaceship is L_0/γ away. It is travelling at v, so it arrive at $t_2 = L_0/\gamma v$.

The problem can be also analyzed using a spacetime diagram. For convenience I have set v = 0.6 c, so $\beta = 0.6$ and $\gamma = 1.25$. The nose is at the origin in both S and S′. The location of the tail (B′) is also shown at t′. Line 1 shows the location of the tail for all time. Note that it is parallel to the ct′ axis. At t′ = 0, a light beam is directed to the left from the nose of the spaceship. The light beam is labelled 3 on the graph. The point where the light beam hits line 1 indicates how long has passed in the S′ reference frame. To read that time correctly, we draw line 2 which is parallel to the x′ axis. The intersection of line 2 and the ct′ axis is t′. Here the value is ct′ = 100 m. According to our answer we found $t' = L_0/c =$ or $ct' = L_0 = 100$ m. So we agree.

The intersection of the light beam and line 1 in the S frame can also be read of the graph at ct = 50. To confirm, recall our answer to (b), $ct_1 = L_0/\gamma(1+\beta) = 100/(1.25)(1.6) = 50$ m.

Note also that we can determine the length of the spaceship is S (L) from the diagram using line 1. Since the spaceship is at rest in frame S′, the length L_0 can always be determine from the origin and any point on line 1. To find L, the length of the moving spaceship, we need to ensure that we measure the ends at the same time in S. Since we know the location of the nose at t=0, we need to find the location of the tail at t=0. That is given by the intersection of line 1 and the x-axis. The length is $L = L_0/\gamma = 80$ m as shown by the arrow marked L.

We can also find out how long it takes the tail to reach the origin in frame S. Line 1 tells us where the tail will be at all time. The intersection of line 1 with the ct axis (i.e. x =0) tells us when the tail is at x = 0, $ct_2 = L_0/\gamma(v/c) = 100/(125)(0.6) = 133$ m.



5. (From French, Special Relativity) Two spaceships, each measuring 100 m in its own rest frame, pass by each other traveling in opposite directions. Instruments on spaceship A determine that the front end of spaceship B requires 5.00×10^{-7} s to traverse the full length of A.

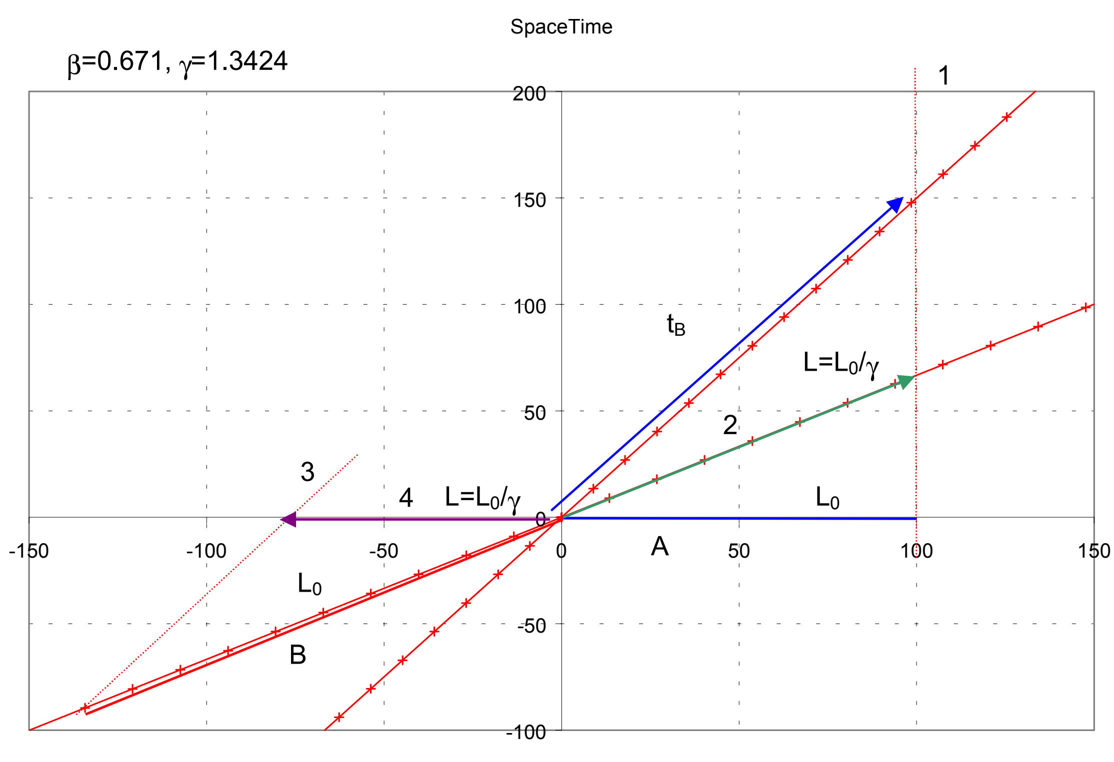
(a) What is the relative velocity of the two spaceships?

(b) A clock of the front end of B reads exactly one o'clock as it passes by the front end of A. What will the clock read as it passes the rear end of A?

(a) The occupant of A determines that the nose of B (and hence the rest of B as well) has moved 100 m in 5.00×10^{-7} s. Since both measurements are in A's frame, A determines the velocity of B to be $v = 100/(5.00 \times 10^{-7}) = 2 \times 10^{8}$ m/s $=0.6671$ c. Note $\gamma = 1.3424$ and $c\Delta t_A = 149.9$ m.

(b) First note that the clock in spaceship A that measures the 5.00×10^{-7} s is not a proper time since the two events do not occur at the same place. The time measured by the clock in B will be a proper time since the clock is in the nose of B for both events. Since we have a proper time, time dilation can be used to relate the two time intervals. The two will be related by $t_A = \gamma t_B$, so $t_B = t_A/\gamma = (5 \times 10^{-7})/01.3424 = 3.725 \times 10^{-7}$ s or $ct_B = 111.7$ m.

The space-time diagram below is used to illustrate the answer. Both ships are shown with their noses at the origin and B moving to the left. Both ships are 100 m long in their own frames. The ct′ axis indicates where the nose of B will be at all times in both frames. The line labelled 1 indicates where the tail of A will be. Where ct′ and 1 intersect, the nose of B is at the tail of A. This is at ct = 150 m and ct′ = 112 m as found earlier. For interest's sake, the length of each ship in the other's frame is shown, vectors 2 and 4. Line 3 is where the tail of B is at all time.



6. Sketch a spacetime diagram showing two spaceships travelling at v = 0.6c, one moving away to the right and one approaching from the left. The receding spaceship sends you a radio signal every 20 m of time by its clock. You send it a radio signal every 20 m of time by your clock. The approaching ship sends you a signal every 40 m of time by its clock and you reciprocate by send a signal to it every 40 m of your time. Sketch these signals on your diagram. Determine how often you see the receding spaceships signal and how often it sees your signals. Repeat for the approaching spaceship.

Let yourself be at the origin of the S frame and let S′ share that origin. Note $\beta = 0.6$ and $\gamma = 1.25$. In the spacetime diagram below, the large dots on the x′ and ct′ axes mark 10 m intervals.

Consider the receding spaceship. The table below shows the times that the spaceship emits the radio signal in it's frame S′ and the events as described in S using the Lorentz Transformation

Event	**S′**	**S**
receding spaceship emits first signal	(0, 0)	(0, 0)
second	(0, 20)	(15, 25)
third	(0, 40)	(30, 50)
fourth	(0, 60)	(45, 75)

These events are every second large dot on the positive ct′ axis. Now each signal has to travel back to you at x = 0, the ct axis, at the speed of light. This is shown by the green arrows; note that the negative slope means moving to left. For the second signal that beam travel from x = 15 to x = 0. That requires t = 15/c or ct = 15 m of extra time. So the second signal arrives to you at 25 + 15 = 40 m. The third signal is emitted at ct = 50 m and needs 30 m to get back for a total of 80 m. So the signals arrive every $c\Delta t = 40$ m. So the arrival rate is half the sending rate.

On the other hand, we can determine the location in S′ that you emit a radio signal.

Event	**S**	**S′**
you send first signal to receding spaceship	(0, 0)	(0, 0)
second	(0, 20)	(-15, 25)
third	(0, 40)	(-30, 50)
fourth	(0, 60)	(-45, 75)

Light-blue dashed lines have been added to the diagram to help the reader find interesting points in the S′ frame. Your second signal must travel from x′ = -15 to the spaceship at x′ = 0, the ct′ axis. This takes t′ = 15/c or ct′ = 15 m. Since you emit the signal ct′ = 25 m (S′ frame) after the ship has left the origin, it arrives at ct′ = 15 + 25 m = 40 m. This is shown by the blue arrows in the upper–right quadrant. The third signal arrives at ct′ = 30 + 50 = 80 m and the fourth at ct′ = 120 m. The interval between the arrival of your signal at the spaceship is ct′ = 40 m. This is the same interval as what you get from the spaceship. You cannot tell which of you is moving.

The location of the approaching spaceship is given by the x′ = 0 axis in the lower-left quadrant. The green arrows moving to the right with slope equal to 1 (speed c) are the radio signals. Where the arrows intersect the ct axis (x = 0) is when you receive the signal. Again we can state the emission events in both frames as shown in the following table.

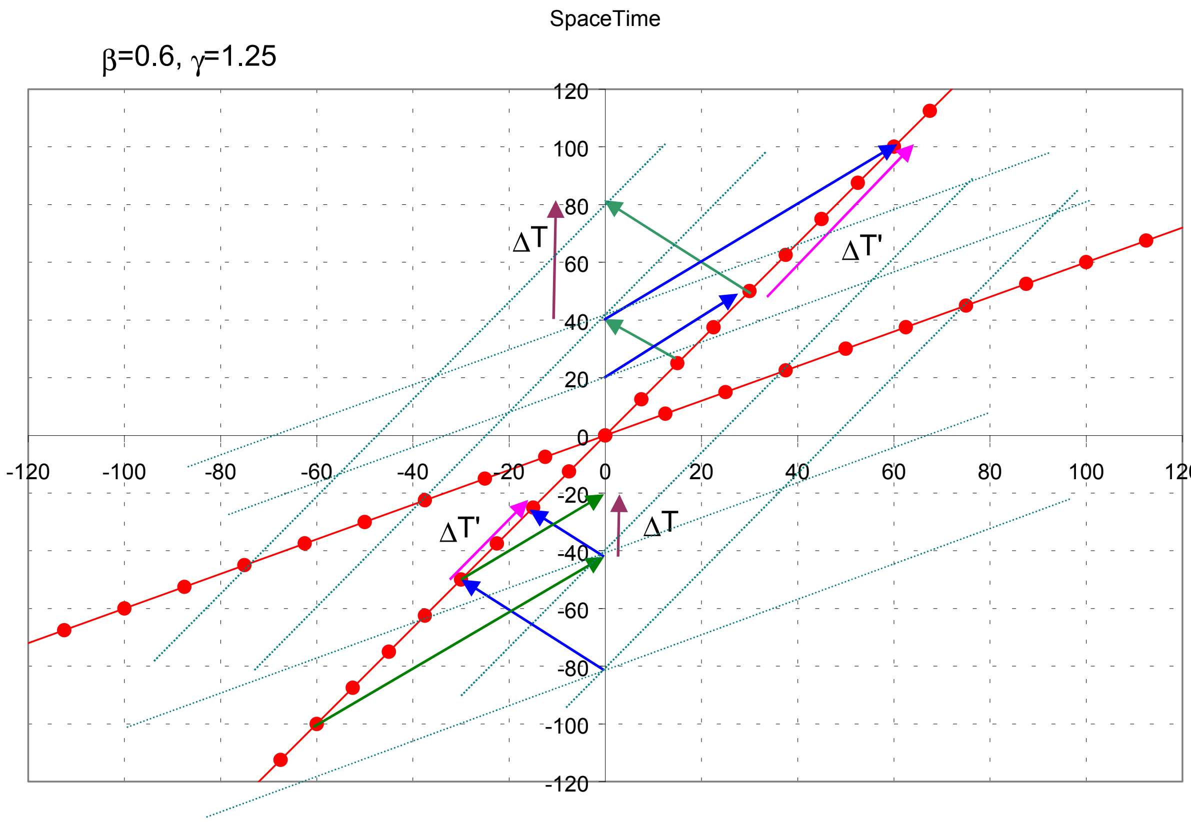
Event	S′	S
approaching spaceship emits first signal	(0, -80)	(-60, -100)
second	(0, -40)	(-30, -50)
third	(0, 0)	(0, 0)

To you the signal is emitted at ct = -100 and has to travel 60 m to reach you. That travel time is t = 60/c or ct = 60 m. So the first signal arrives at ct = -100 + 60 = 40 m. Similarly the second signal arrives at ct = -50 + 30 = 20 m. You see the signals as arriving every 20 m. So the arrival rate is double the sending rate.

You also send signals to the approaching ship and we can determine the location in S′ that you emit a radio signal.

Event	S	S′
you send first signal to approaching spaceship	(0, -80)	(60, -100)
second	(0, -40)	(30, -50)
third	(0, 00)	(0, 0)

Your first signal must travel from x′ = 60 to the spaceship at x′ = 0, the ct′ axis. This takes t′ = 60/c or ct′ = 60 m. Since you emit the signal ct′ = -100 m (S′ frame) before the ship arrives at the origin, it arrives at ct′ = -100 + 60 m = -40 m. This is shown by the blue arrows in the lower–left quadrant. The second signal arrives at ct′ = -50 + 30 = 20 m. The interval between the arrival of your signal at the spaceship is ct′ = 20 m. This is the same interval as what you get from the spaceship. Since they are the same, you cannot tell which of you is moving.


SpaceTime
β=0.6, γ=1.25
ΔT
ΔT'
ΔT'
ΔT
120
100
80
60
40
20
0
-20
-40
-60
-80
-100
-120
-120
-100
-80
-60
-40
-20
0
20
40
60
80
100
120

7. Here is a relativity paradox. You are floating in space beside the nose of a space ship that is two lightyears (ly) long. The ship accelerates from zero to .866 c in 10 seconds. Due to length contraction, the ship is now only 1.00 ly long (in your frame of reference). This means that the tail of the ship has moved 1.00 ly in 10 s (in your frame of reference), thus has an average speed of three million times the speed of light. Try to resolve this paradox or at least point out some of the problems.

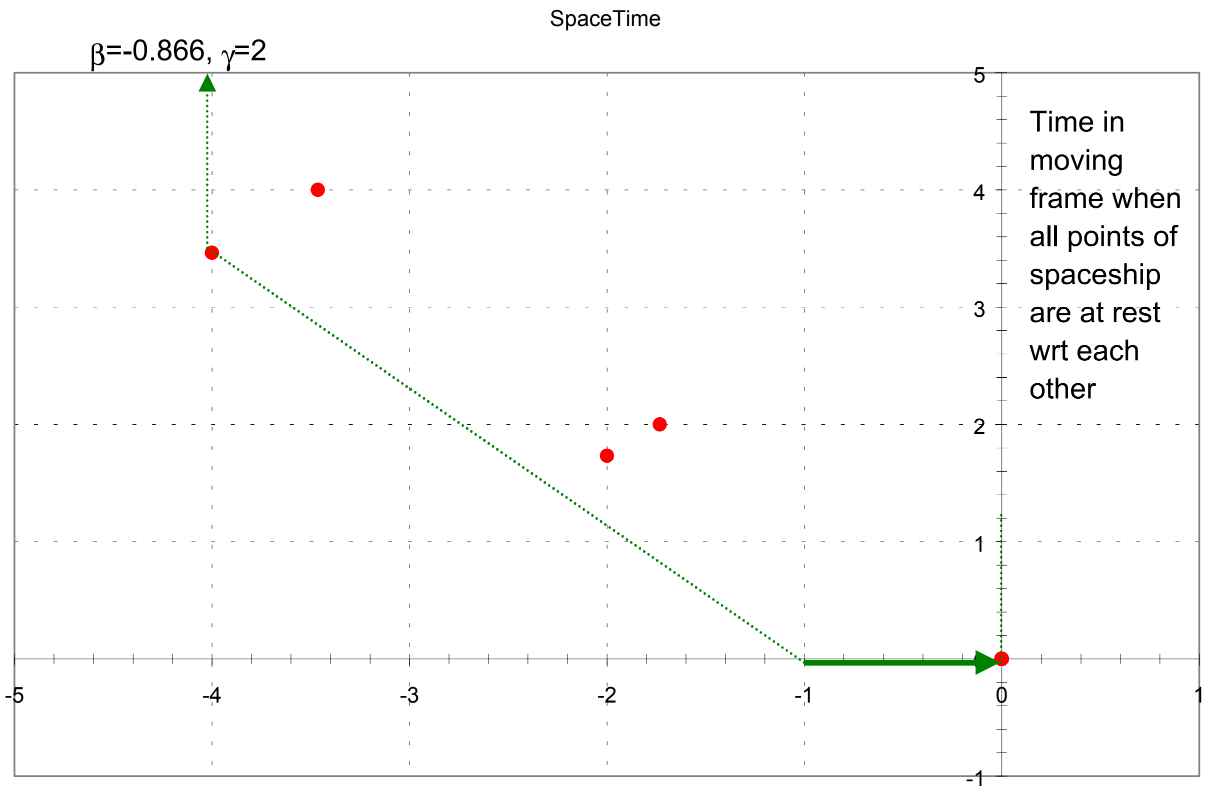
Let's assume that there is only one engine at the tail. When it starts, it will take the nose two years to find out about it and also start moving since information and objects cannot exceed the speed of light. So this cannot the case described. So let's say there are many identical engines, with identical amounts of fuel all along the length of the ship. We can easily have all the stationary engines synchronized to start off at the same instant, since they will all initially be in the same stationary frame S. By having identical engines and amounts of fuel, we expect all the pieces to end up travelling at 0.866 c together. That expectation is the crux of the paradox. What is simultaneous in one frame is not simultaneous in another.

Lets consider the nose at (0,0) and the tail at (-2 ly, 0) firing at the same time. In a frame travelling at v = 0.866 c (S′), $\gamma = 2$ and the nose is at (0, 0) and the tail is at (-4 ly, +3.46 y). Now those engines are still identical and carry the same amount of fuel in the new frame. In S′, however, they don't fire at the same time. Physically we expect this to subject the ship to enormous stresses that rip it apart. If by some miracle that doesn't happen, an observer in S′ sees that the ends doesn't reach the same terminal velocity until 3.46 years have passed.

So the assumption that it takes only 10 s (in frame S) to come to rest in frame S′ is incorrect. Since it is not at rest in S′ and 2 ly long, you do not see it length-contracted to 1 ly, you do not see the tail exceeding the speed of light.

It helps to study the spacetime diagram for this problem from the point of view of an observer moving at 0.866 c, S′. The origin in this frame is the nose of the spaceship as it just starts to move. The scale of graph is in years and lightyears. Notice it has a rectangular grid. From the point of view of the observer in this frame he is stationary, and you in your spacesuit are moving to the left at 0.866 c. In this frame the length of the spaceship is length contracted to 1 ly. Remember that the 2-ly length was for the ship at rest your frame and thus was a proper length. The length dilation formula says $L = L_P/\gamma = (2\text{ ly})/2 = 1$ ly. The wordlines of the nose, midpoint, and tail are shown. Since the scale is years, the 10 s acceleration (in S) is a sharp change in direction. The nose stays at the origin (just about) and moves with the observer. This is shown by the dashed green arrow on the ct′-axis. From the observer in S′, the tail, 1 ly behind the nose, moves to the left at 0.866 c. When the tail is 4 ly behind the observer, some 3.46 y later, it suddenly accelerates to a complete stop with respect to the observer. The blue dashed horizontal line indicates the time when all points on the spaceship are at rest with the observer. The spaceship has stretched from 1 ly to 4 ly long from this observer's point of view. So he certainly would have expected the stresses to rip the ship apart. Note that the 4 ly is a proper length since it is measured at the same time in S′.

The red lines on the diagram are for your frame moving to the left at 0.866 c with respect to the observer in S′. The large dots mark off 1 year intervals. Note that the length you see for the spaceship at t = 0 is the heavy red arrow which is 2 ly long. When the ship is finally at rest in the observer's frame you still see it as 2 ly long. That means you did not see the tail exceed the speed of light. It also means that since you expected to see its length as 1 ly, that it must have stretched or more-likely ripped itself apart.


SpaceTime
β=-0.866, γ=2
Time in moving frame when all points of spaceship are at rest wrt each other
5
4
3
2
1
-1
-5
-4
-3
-2
-1
0
1