IMPACT

Today's Objectives:

Students will be able to:

1. Understand and analyze the mechanics of impact.
2. Analyze the motion of bodies undergoing a collision, in both central and oblique cases of impact.

Concepts

- Central Impact
- Coefficient of Restitution
- Oblique Impact

READING QUIZ

1. When the motion of one or both of the particles is at an angle to the line of impact, the impact is said to be _____.

   A) central impact
   B) oblique impact
   C) major impact
   D) None of the above.

2. The ratio of the restitution impulse to the deformation impulse is called _______.

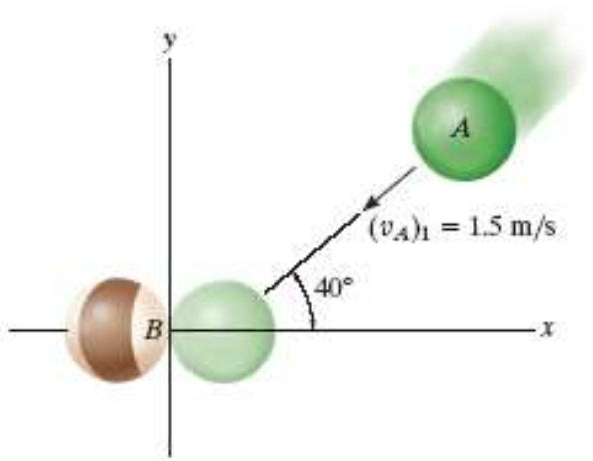
   A) impulse ratio
   B) restitution coefficient
   C) energy ratio
   D) mechanical efficiency

APPLICATIONS

The quality of a tennis ball is measured by the height of its bounce. This can be quantified by the coefficient of restitution of the ball.

If the height from which the ball is dropped and the height of its resulting bounce are known, how can we determine the coefficient of restitution of the ball?

APPLICATIONS
(continued)

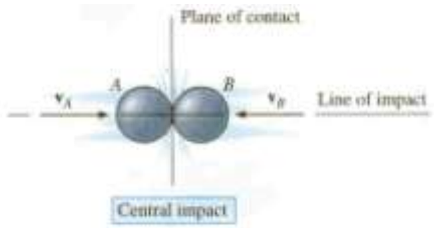


In a game of billiards, it is important to be able to predict the trajectory and speed of a ball after it is struck by another ball.

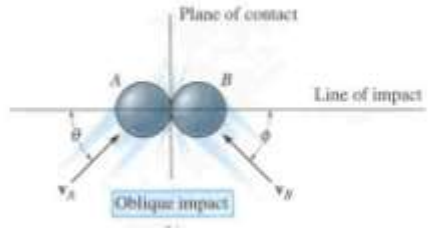
If we know the velocity of ball A before the impact, how can we determine the magnitude and direction of the velocity of ball B after the impact?

IMPACT
(Section 15.4)

Impact occurs when two bodies collide during a very short time period, causing large impulsive forces to be exerted between the bodies. Common examples of impact are a hammer striking a nail or a bat striking a ball. The line of impact is a line through the mass centers of the colliding particles. In general, there are two types of impact:



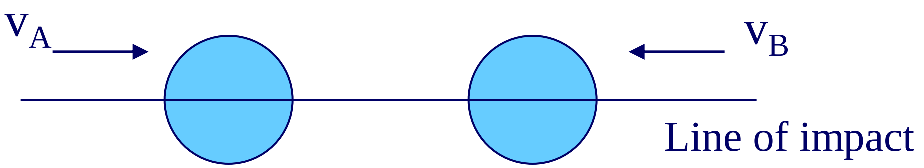
Central impact occurs when the directions of motion of the two colliding particles are along the line of impact.



Oblique impact occurs when the direction of motion of one or both of the particles is at an angle to the line of impact.

CENTRAL IMPACT

Central impact - velocities of the two objects are along the line of impact (recall that the line of impact is a line through the particles' mass centers).



Once the particles contact, they may deform if they are non-rigid. In any case, energy is transferred between the two particles.

There are two primary equations used when solving impact problems.

CENTRAL IMPACT
(continued)

In most problems, the initial velocities of the particles, $(v_A)_1$ and $(v_B)_1$, are known, and it is necessary to determine the final velocities, $(v_A)_2$ and $(v_B)_2$.

First use is the conservation of linear momentum, applied along the line of impact.

$$(m_A\, v_A)_1 + (m_B\, v_B)_1 = (m_A\, v_A)_2 + (m_B\, v_B)_2$$

This provides one equation, but there are usually two unknowns, $(v_A)_2$ and $(v_B)_2$. So another equation is needed. The principle of impulse and momentum is used to develop this equation, which involves the coefficient of restitution, or *e*.

CENTRAL IMPACT
(continued)

If KE was conserved in a 1D collision, KE lost by particle A equals KE gained by B.

$$\tfrac{1}{2}m_A v_{A2}^2 - \tfrac{1}{2}m_A v_{A1}^2 = -\left(\tfrac{1}{2}m_B v_{B2}^2 - \tfrac{1}{2}m_B v_{B1}^2\right)$$

$$m_A v_{A2}^2 - m_A v_{A1}^2 = -\left(m_B v_{B2}^2 - m_B v_{B1}^2\right)$$

$$m_A(v_{A2} - v_{A1})(v_{A2} + v_{A1}) = -m_B(v_{B2} - v_{B1})(v_{B2} + v_{B1})$$

$$v_{A2} + v_{A1} = v_{B2} + v_{B1} \quad \text{Since } I_1 = -I_2$$

$$\text{Or} \qquad v_{A2} - v_{B2} = -(v_{A1} - v_{B1})$$

Relative separation velocities are same after collision.

CENTRAL IMPACT
(continued)

In general KE is not conserved. So

$$v_{A2} - v_{B2} < -(v_{A1} - v_{B1}) \; .$$

The equation defining the coefficient of restitution, *e*, is

$$\boxed{v_{A2} - v_{B2} = -e(v_{A1} - v_{B1})}$$

If a value for *e* is specified, this relation provides the second equation necessary to solve for v_{A2} and v_{B2}.

COEFFICIENT OF RESTITUTION

In general, e has a value between zero and one. The two limiting conditions can be considered:

- Elastic impact (e = 1): In a perfectly elastic collision, no energy is lost and the relative separation velocity equals the relative approach velocity of the particles. In practical situations, this condition cannot be achieved.

- Plastic impact (e = 0): In a plastic impact, the relative separation velocity is zero. The particles stick together and move with a common velocity after the impact.

Some typical values of e are:

Steel on steel: 0.5 – 0.8
Wood on wood: 0.4 – 0.6
Lead on lead: 0.12 – 0.18
Glass on glass: 0.93 – 0.95

IMPACT: ENERGY LOSSES

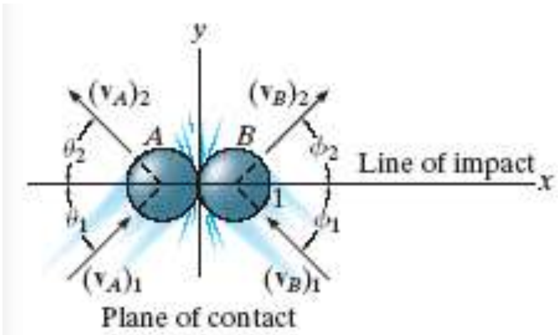
Once the particles' velocities before and after the collision have been determined, the energy loss during the collision can be calculated on the basis of the difference in the particles' kinetic energy. The energy loss is

$$\sum U_{1\text{-}2} = \sum T_2 - \sum T_1 \quad \text{where } T_i = 0.5 m_i (v_i)^2$$

During a collision, some of the particles' initial kinetic energy will be lost in the form of heat, sound, or due to localized deformation.

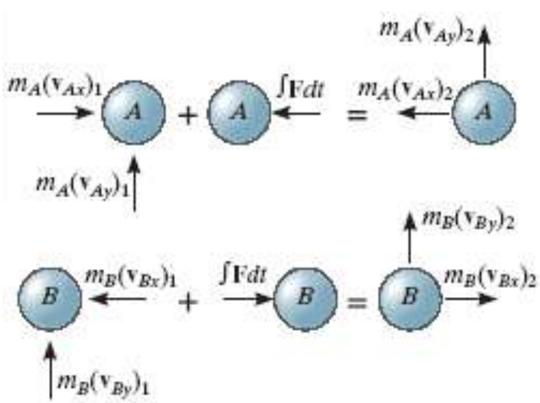
In a plastic collision ($e = 0$), the energy lost is a maximum, although it does not necessarily go to zero. Why?

OBLIQUE IMPACT



In an oblique impact, one or both of the particles' motion is at an angle to the line of impact. Typically, there will be four unknowns: the magnitudes and directions of the final velocities.

The four equations required to solve for the unknowns are:



Conservation of momentum and the coefficient of restitution equation are applied along the line of impact (x-axis):

$$m_A(v_{Ax})_1 + m_B(v_{Bx})_1 = m_A(v_{Ax})_2 + m_B(v_{Bx})_2$$

$$e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$$

Momentum of each particle is conserved in the direction perpendicular to the line of impact (y-axis):

$$m_A(v_{Ay})_1 = m_A(v_{Ay})_2 \quad \text{and} \quad m_B(v_{By})_1 = m_B(v_{By})_2$$

PROCEDURE FOR ANALYSIS

- In most impact problems, the initial velocities of the particles and the coefficient of restitution, *e*, are known, with the final velocities to be determined.

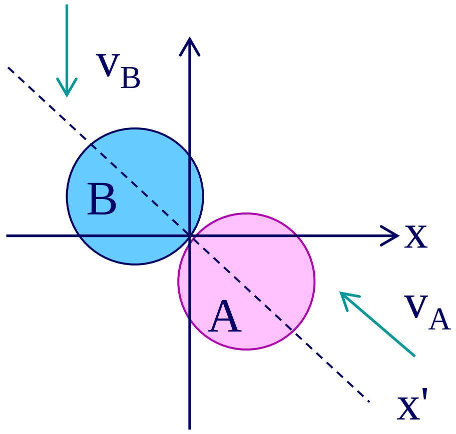
- Define the x-y axes. Typically, the x-axis is defined along the line of impact and the y-axis is in the plane of contact perpendicular to the x-axis.

- For both central and oblique impact problems, the following equations apply along the line of impact (x-dir.):
$\sum m(v_x)_1 = \sum m(v_x)_2$ and $e = [(v_{Bx})_2 - (v_{Ax})_2]/[(v_{Ax})_1 - (v_{Bx})_1]$

- For oblique impact problems, the following equations are also required, applied perpendicular to the line of impact (y-dir.):

$$m_A(v_{Ay})_1 = m_A(v_{Ay})_2 \quad \text{and} \quad m_B(v_{By})_1 = m_B(v_{By})_2$$

EXAMPLE



Given: A 1-kg disk A travelling at $v_A = 4$ m/s at 45° as shown hits 2-kg disk B travelling at $v_B = 2$ m/s . The coefficient of restitution is $e = 0.6$.

Find: The velocity of each disk after the collision.

Plan: 1) Find v_A' and v_B' in the x' frame.*

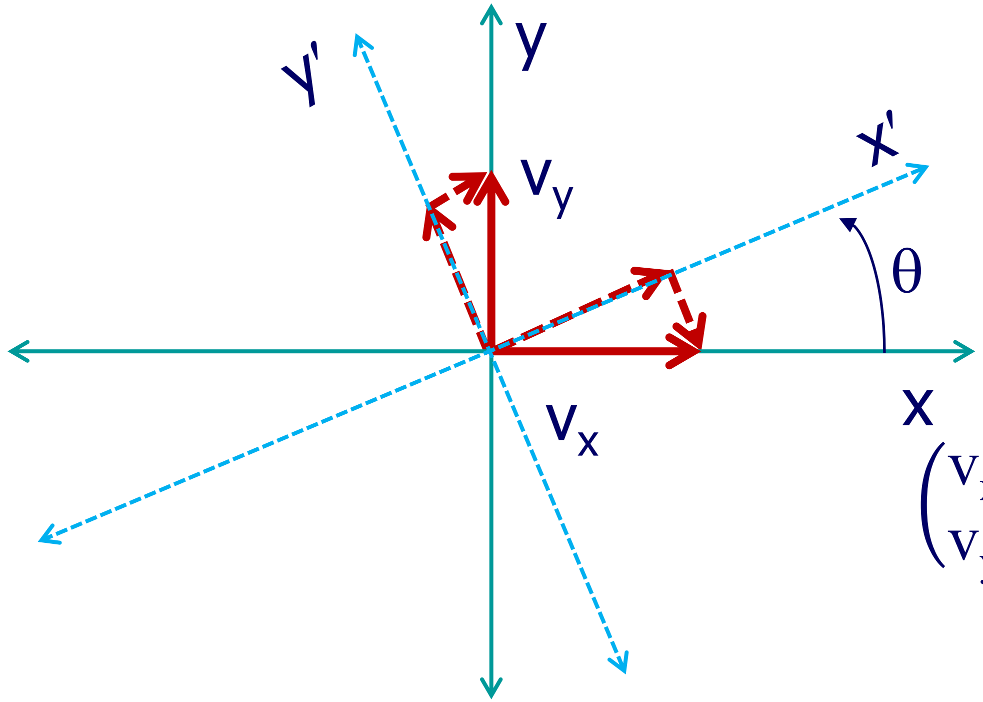
2) Use CofL and restitution equations along x axis for v_{Ax}' and v_{Bx}'.

3) v_{Ay}' and v_{By}' never change since there is no $F_{y'}$.

4) Find v_{Af} and v_{Bf} in the orignal x frame.*

**Finding a velocity in a rotated frame can be done using a rotation matrix.*

Rotation Matrix for Frame Rotation



$$v_x' = v_x\cos\theta + v_y\sin\theta$$

$$v_y' = -v_x\sin\theta + v_y\cos\theta$$

$$\downarrow$$

$$\begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$\downarrow$$

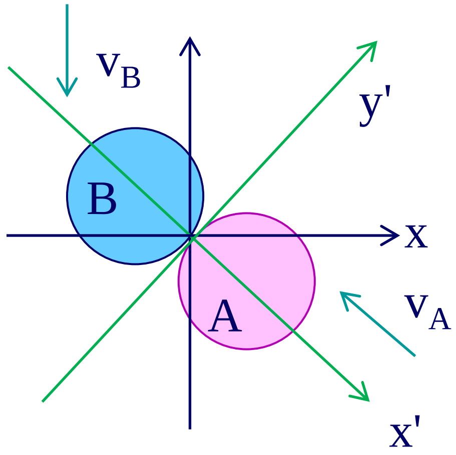
$$\begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = R(\theta) \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

After finding final velocities need to convert back

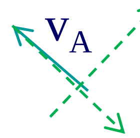
$$\begin{pmatrix} v_{xf} \\ v_{yf} \end{pmatrix} = R(-\theta) \begin{pmatrix} v_{xf}' \\ v_{yf}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_{xf}' \\ v_{yf}' \end{pmatrix}$$

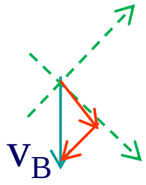
Note: Rotating a frame is not the same as rotating a vector.

EXAMPLE



- Make sure x' is thru CMs
- θ measured from +x axis (0°)
- Here θ = –45°





$$\begin{pmatrix} v_{Ax}' \\ v_{Ay}' \end{pmatrix} = R(-45°)\begin{pmatrix} v_{Ax} \\ v_{Ay} \end{pmatrix} = \begin{pmatrix} \cos(-45°) & \sin(-45°) \\ -\sin(-45°) & \cos(-45°) \end{pmatrix}\begin{pmatrix} -4\cos(45°) \\ +4\sin(45°) \end{pmatrix} = \begin{pmatrix} -4 \\ \mathbf{0} \end{pmatrix}$$

$$\begin{pmatrix} v_{Bx}' \\ v_{By}' \end{pmatrix} = R(-45°)\begin{pmatrix} v_{Bx} \\ v_{By} \end{pmatrix} = \begin{pmatrix} \cos(-45°) & \sin(-45°) \\ -\sin(-45°) & \cos(-45°) \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1.4142 \\ -\mathbf{1.4142} \end{pmatrix}$$

Note: Good to check signs are consistent!

Note: In the y' direction, no force. So v_{Ay}' and v_{By}' do not change.

EXAMPLE
(continued)

For collision in x'

CL: $(1)\, v_{Axf}' + (2)\, v_{Bxf}' = (1)\,(-4) + (2)(1.4142) = -1.1716$

e: $v_{Axf}' - v_{Bxf}' = (0.6)\,(1.4142 - -4) = 3.2485$

Solving: $v_{Axf}' = 1.7752$ and $v_{Bxf}' = -1.4734$, or

$$\mathbf{v}_{Af}' = \begin{pmatrix} 1.7752 \\ \mathbf{0} \end{pmatrix} \text{ and } \mathbf{v}_{Bf}' = \begin{pmatrix} -1.4734 \\ -\mathbf{1.4142} \end{pmatrix}.$$

Note: $v_{Af}' = 1.7752$ and $v_{Bf}' = 2.0423$.

EXAMPLE
(continued)

Back in the original frame, a +45° rotation:

$$\begin{pmatrix} v_{Axf} \\ v_{Ayf} \end{pmatrix} = R(+45°)\begin{pmatrix} v_{Axf}' \\ v_{Ayf}' \end{pmatrix} = \begin{pmatrix} \cos(+45°) & \sin(+45°) \\ -\sin(+45°) & \cos(+45°) \end{pmatrix}\begin{pmatrix} 1.7752 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.2552 \\ -1.2552 \end{pmatrix}$$

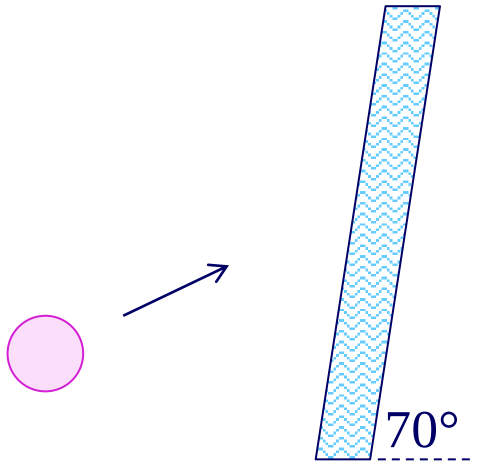
$$\begin{pmatrix} v_{Bxf} \\ v_{Byf} \end{pmatrix} = R(+45°)\begin{pmatrix} v_{Bxf}' \\ v_{Byf}' \end{pmatrix} = \begin{pmatrix} \cos(+45°) & \sin(+45°) \\ -\sin(+45°) & \cos(+45°) \end{pmatrix}\begin{pmatrix} -1.4734 \\ -1.4142 \end{pmatrix} = \begin{pmatrix} -2.0418 \\ 0.0418 \end{pmatrix}$$

Note: $v_{Af} = 1.7752$ and $v_{Bf} = 2.0423$. Magnitude of a vector same in rotated frames.

Collision with a Wall

- Cannot conserve momentum with a fixed object. Why?
- Can still use equation of restitution!

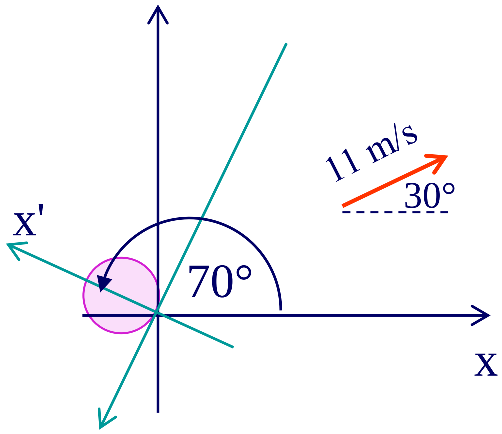
EXAMPLE



Given: A 0.300-g ball travelling at v = 11 m/s at 30° a.h. hits a wall oriented as shown. The coefficient of restitution is $e = 0.4$.

Find: The velocity of the ball after the collision.

EXAMPLE
(continued)



- Normal is perpendicular to surface
- θ measured from +x axis (0°)
- Here θ = 90° + 70° = 160°

In the new frame:

$$\begin{pmatrix} v_x' \\ v_y' \end{pmatrix} = R(160°)\begin{pmatrix} v_A \\ v_A \end{pmatrix} = \begin{pmatrix} \cos(160°) & \sin(160°) \\ -\sin(160°) & \cos(160°) \end{pmatrix}\begin{pmatrix} 11\cos(30°) \\ 11\sin(30°) \end{pmatrix} = \begin{pmatrix} -7.0707 \\ \mathbf{-8.4265} \end{pmatrix}$$

Note: In the y' direction, no force. So v_y' does not change.

For collision in x' with the stationary wall

e: $$v_{xf}' - \cancelto{0}{v_{wall\ xf}'} = -(0.4)\,(-7.0707 - \cancelto{0}{v_{wall\ xf}'}) = 2.8283$$

or, $v_{xf}' = 2.8283$

EXAMPLE
(continued)

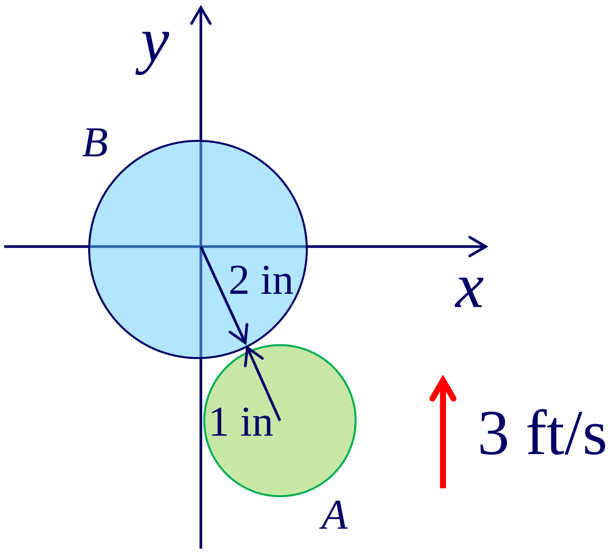
Or, $\mathbf{v}_f{}' = \begin{pmatrix} v_{xf}{}' \\ v_{yf}{}' \end{pmatrix} = \begin{pmatrix} 2.8283 \\ \mathbf{-8.4265} \end{pmatrix}$

Back in the original frame, a $-160°$ rotation:

$$\begin{pmatrix} v_{xf} \\ v_{yf} \end{pmatrix} = R(-160°)\begin{pmatrix} v_{xf}{}' \\ v_{yf}{}' \end{pmatrix} = \begin{pmatrix} \cos(-160°) & \sin(-160°) \\ -\sin(-160°) & \cos(-160°) \end{pmatrix}\begin{pmatrix} 2.8283 \\ -8.4265 \end{pmatrix} = \begin{pmatrix} 0.2243 \\ 8.8856 \end{pmatrix}$$

The final velocity of the ball is $v_f = 8.8885$ m/s at 88.6° a.h.

EXAMPLE



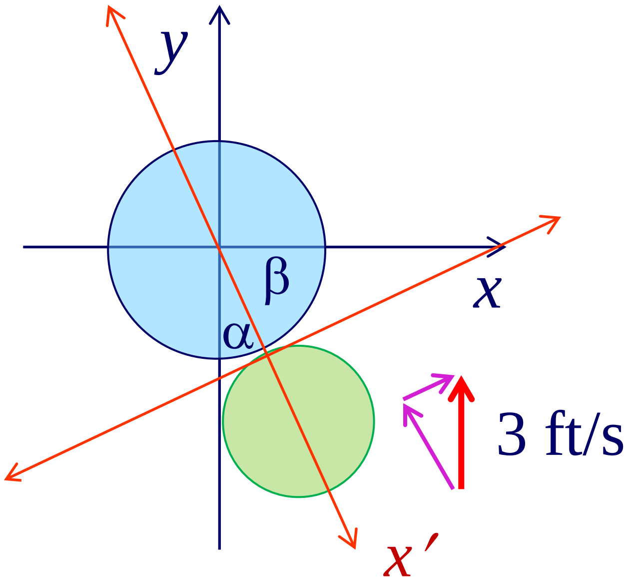
Given: Disk *A* weighs 2 lb and is moving at 3 ft/s. Disk *B* weighs 11 lb and is initially at rest. The coefficient of restitution is $e = 0.6$.

Find: The velocities after the collision.

Plan: The radii can be used to make a right triangle and determine the angle needed for rotating the axes.

The angle with the negative *y* axis is $\alpha = \arcsin(1/3) = 19.471°$.
The angle with the positive *x* axis is $\beta = \arccos(1/3) = 70.529°$.

EXAMPLE (cont'd)



Must first decide where the positive x' axis is.

Let's take θ = –β. Note that CCW is + and CW is –.

You should pay attention to the components of each velocity vector in the new coordinate system.

EXAMPLE (cont'd)

$$\mathbf{v}_A = \boldsymbol{j}3\text{ ft/s},\ \mathbf{v}_B = 0 \qquad \theta = -70.529°$$

$$\begin{pmatrix} v_{Ax}' \\ v_{Ay}' \end{pmatrix} = \begin{pmatrix} \cos(-70.529°) & \sin(-70.529°) \\ -\sin(-70.529°) & \cos(-70.529°) \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2.82843 \\ 1 \end{pmatrix}$$

Note $\mathbf{v}_B' = 0$. Also y' is $\perp$ to collision axis so $v_{Ayf}' = 1$ & $v_{Byf}' = 0$

CM: $(2)v_{Axf}' + (11)v_{Bxf}' = (2)(-2.82843) + (11)(0) = -5.65685$

e: $v_{Axf}' - v_{Bxf}' = -(0.6)\,(-\,2.82843 - 0) = 1.6971$

$$\begin{pmatrix} 2 & 11 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{Axf}' \\ v_{Bxf}' \end{pmatrix} = \begin{pmatrix} -5.65685 \\ 1.6971 \end{pmatrix} \Rightarrow \begin{pmatrix} v_{Axf}' \\ v_{Bxf}' \end{pmatrix} = \begin{pmatrix} 1.00083 \\ -0.69623 \end{pmatrix}$$

So $\mathbf{v}_{Af}' = \begin{pmatrix} 1.00083 \\ 1 \end{pmatrix}$ & $\mathbf{v}_{Bf}' = \begin{pmatrix} -0.69623 \\ 0 \end{pmatrix}$

EXAMPLE (cont'd)

Next, rotate back to the original axes for final answer.

$$\begin{pmatrix} v_{Axf} \\ v_{Ayf} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1.00083 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.27642 \\ -0.61026 \end{pmatrix}$$

$$\begin{pmatrix} v_{Bxf} \\ v_{Byf} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} -0.69623 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.23208 \\ 0.65461 \end{pmatrix}$$

CONCEPT QUIZ

1. Two balls impact with a coefficient of restitution of 0.79. Can one of the balls leave the impact with a kinetic energy greater than before the impact?

   A) Yes

   B) No

   C) Impossible to tell

   D) Don't pick this one!

2. Under what condition is the energy lost during a collision maximum?

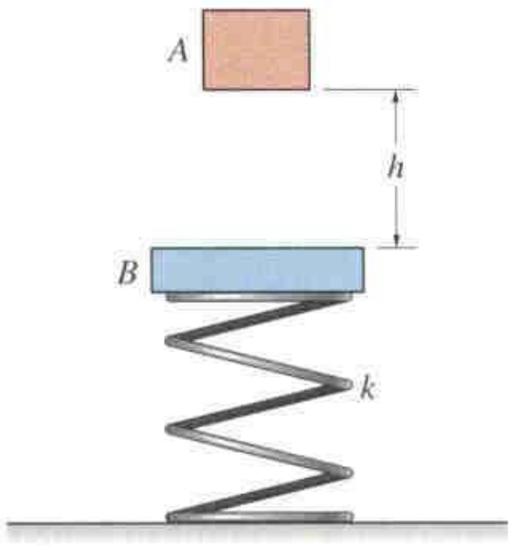
   A) $e = 1.0$

   B) $e = 0.0$

   C) $e = -1.0$

   D) Collision is non-elastic.

GROUP PROBLEM SOLVING



Given: A 2 kg block A is released from rest, falls a distance h = 0.5 m, and strikes plate B (3 kg mass). The coefficient of restitution between A and B is e = 0.6, and the spring stiffness is k = 30 N/m.

Find: The velocity of block A just after the collision.

Plan:
1) Determine the speed of the block just before the collision using projectile motion or an energy method.
2) Analyze the collision as a central impact problem.

GROUP PROBLEM SOLVING
(continued)

Solution:

1) Determine the speed of block A just before impact by using conservation of energy. Defining the gravitational datum at the initial position of the block ($h_1 = 0$) and noting the block is released from rest ($v_1 = 0$):

$$T_1 + V_1 = T_2 + V_2$$

$$0.5m(v_1)^2 + mgh_1 = 0.5m(v_2)^2 + mgh_2$$

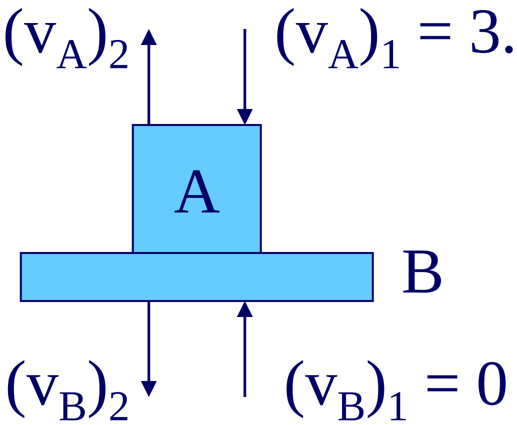
$$0 + 0 = 0.5(2)(v_2)^2 + (2)(9.81)(-0.5)$$

$$v_2 = 3.132 \text{ m/s} \downarrow$$

This is the speed of the block just before the collision. Plate (B) is at rest, velocity of zero, before the collision.

GROUP PROBLEM SOLVING
(continued)

2) Analyze the collision as a central impact problem.



Apply conservation of momentum to the system in the vertical direction:

$$+\uparrow \quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$$

$$(2)(-3.132) + 0 = (2)(v_A)_2 + (3)(v_B)_2$$

Using the coefficient of restitution:

$$+\uparrow \quad e = [(v_B)_2 - (v_A)_2]/[(v_A)_1 - (v_B)_1]$$

$$\Rightarrow \quad 0.6 = [(v_B)_2 - (v_A)_2]/[-3.132 - 0] \quad \Rightarrow \quad -1.879 = (v_B)_2 - (v_A)_2$$

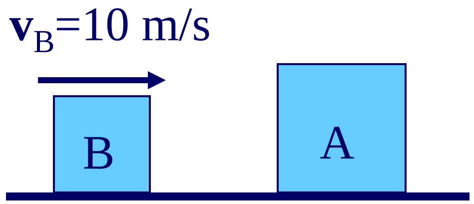
Solving the two equations simultaneously yields

$$(v_A)_2 = -0.125 \text{ m/s} \downarrow, \; (v_B)_2 = -2.00 \text{ m/s} \downarrow$$

Both the block and plate will travel down after the collision.

ATTENTION QUIZ

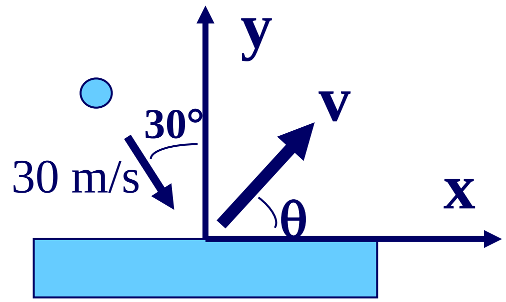
1. Block B (1 kg) is moving on the smooth surface at 10 m/s when it squarely strikes block A (3 kg), which is at rest. If the velocity of block A after the collision is 4 m/s to the right, $(v_B)_2$ is



   A) 2 m/s $\rightarrow$ B) 7 m/s $\leftarrow$

   C) 7 m/s $\rightarrow$ D) 2 m/s $\leftarrow$

2. A particle strikes the smooth surface with a velocity of 30 m/s. If e = 0.8, $(v_x)_2$ is _____ after the collision.



   A) zero B) equal to $(v_x)_1$

   C) less than $(v_x)_1$ D) greater than $(v_x)_1$