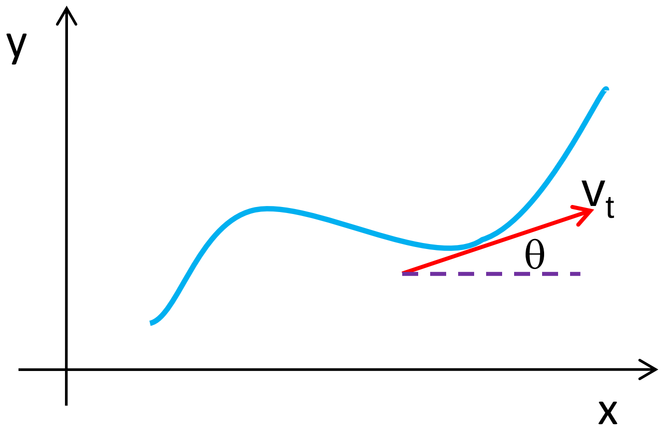
2D Paths and Direction
Curvilinear Unit Vectors



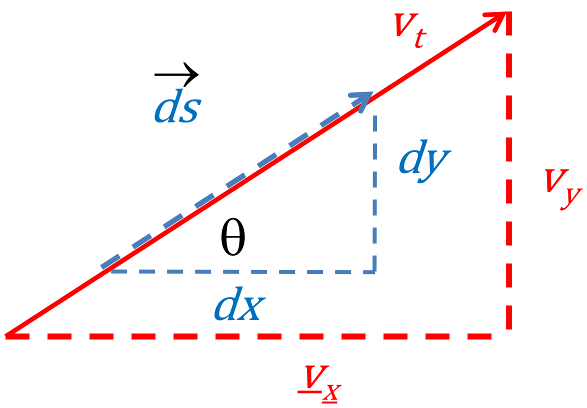
Can we find a unit vector that points in the direction of ***v***? Can we find θ?

If we know v_x and v_y

$$\vec{v} = \hat{i}v_x + \hat{j}v_y$$

Then $\hat{u}_T = \frac{\vec{v}}{v} = \frac{\hat{i}v_x + \hat{j}v_y}{\sqrt{v_x^2 + v_y^2}}$ And $tan\theta = \frac{v_y}{v_x}$

If we know $y = f(x)$



Note $\vec{v_t}$ and $\vec{ds}$ (exaggerated) are parallel

$$tan\theta = \frac{v_y}{v_x} = \frac{dy/_{dt}}{dx/_{dt}} = \frac{dy}{dx}$$

$$\vec{v}_t = \hat{\imath} v_t cos\theta + \hat{\jmath} v_t sin\theta$$

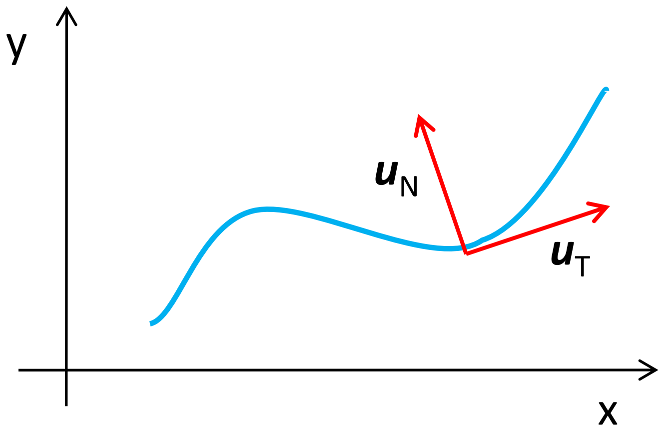
$$\text{So } \hat{u}_t = \hat{\imath} cos\theta + \hat{\jmath} sin\theta$$

Example. Curve is $y = 0.1x^2$ and v is a constant 5 m/s. What is velocity at x = 2 m?

$tan\theta = 2(0.1)(2) = 0.4$, so $\theta = 21.80°$

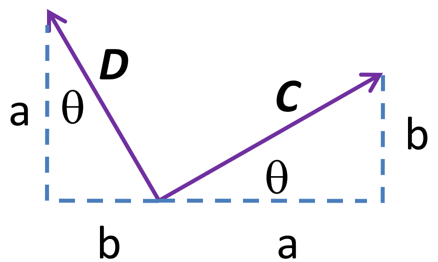
$$\hat{u}_t = \hat{\imath} cos(21.80°) + \hat{\jmath} sin(21.80°)$$

Can we find a unit vector $\boldsymbol{u}_N$?



Note if $\boldsymbol{C} = \boldsymbol{i}a + \boldsymbol{j}b$, and $\boldsymbol{D} = -\boldsymbol{i}b + \boldsymbol{j}a$.

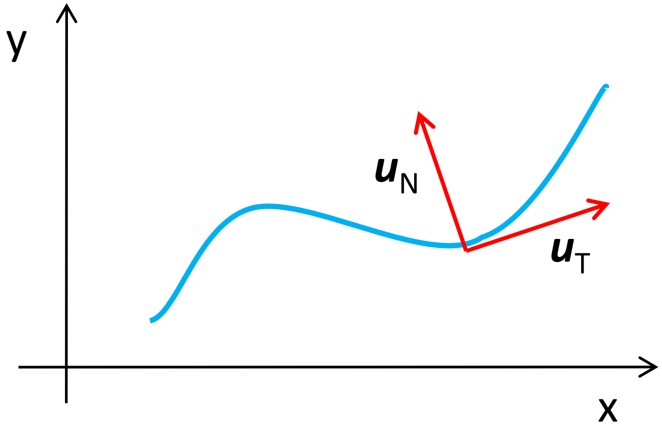
Then $\boldsymbol{C} \bullet \boldsymbol{D} = -ab + ba = 0$

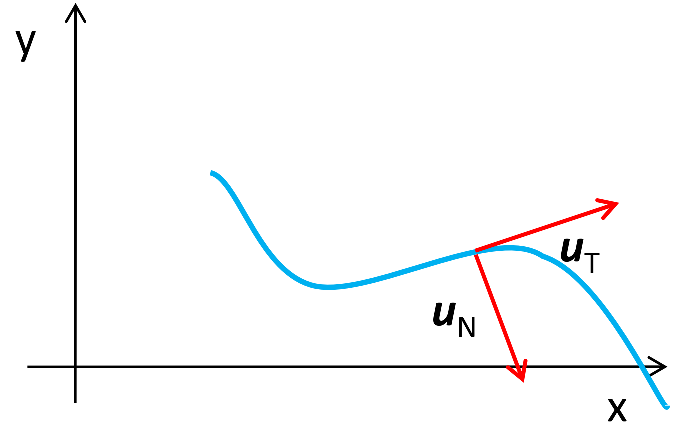


So if $\hat{u}_T = \dfrac{\hat{i}v_x + \hat{j}v_y}{\sqrt{v_x^2 + v_y^2}}$, then $\hat{u}_N = \dfrac{-\hat{i}v_y + \hat{j}v_x}{\sqrt{v_x^2 + v_y^2}}$

Or $\hat{u}_t = \hat{\imath}cos\theta + \hat{\jmath}sin\theta$, and $\hat{u}_n = -\hat{\imath}sin\theta + \hat{\jmath}cos\theta$

Warning – normals depend on curvature

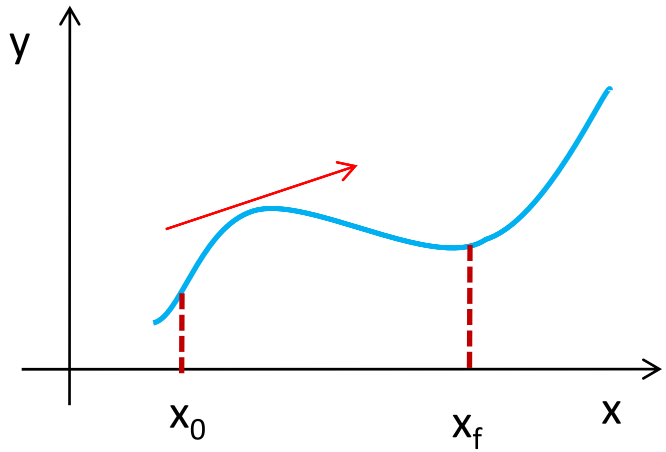




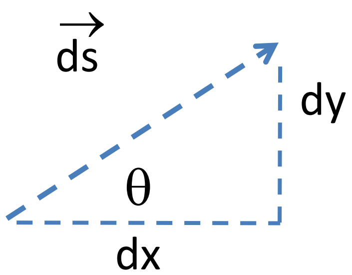
$$\hat{u}_N = \frac{-\hat{i}v_y + \hat{j}v_x}{\sqrt{v_x^2 + v_y^2}}$$

$$\hat{u}_N = \frac{\hat{i}v_y - \hat{j}v_x}{\sqrt{v_x^2 + v_y^2}}$$

2D Paths and Path Length



$$S = \int_{start}^{end} ds$$



$$d\vec{s} = \hat{i}\,dx + \hat{j}\,dy$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= dx\sqrt{1 + (dy/dx)^2}$$

$$S = \int_{x_0}^{x_f} dx\sqrt{1 + (dy/dx)^2}$$

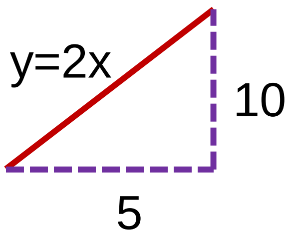
2D Paths and Path Length
Example 1

If y = 2x, find the length s of the curve between x = 0 and x = 5.

$$s = \int_0^5 \sqrt{1 + \frac{dy}{dx}^2}\, dx = \int_0^5 \sqrt{1 + (2)^2} dx$$

$$= \sqrt{5}\, x \Big|_0^5 = 5\sqrt{5}$$

Of course, this was a straight line and we could have used Pythagoras to get the result.



2D Paths and Path Length
Example 2

If y = 2x, find s as a function of time if x = 0 at t = 0 and if the horizontal velocity is v_x = 3t. Find the tangential speed v_t and acceleration.

$$s = \int_0^x \sqrt{1 + \frac{dy}{dx}^2}\, dx = \int_0^x \sqrt{1 + (2)^2} dx = \sqrt{5}\, x \Big|_0^x = \sqrt{5}x$$

$$dx = v_x dt \qquad \int_0^x dx = \int_0^t 3t\, dt \qquad x = \tfrac{3}{2}t^2 \qquad s = \tfrac{3\sqrt{5}}{2}t^2$$

$$v_t = \frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \sqrt{5}v_x = 3\sqrt{5}t$$

$$a_t = \frac{dv_t}{dt} = 3\sqrt{5} \qquad a_n = \frac{v_t^2}{\rho} = \frac{v_t^2}{\infty} = 0$$