CURVILINEAR MOTION: CYLINDRICAL COMPONENTS

Today's Objectives:

Students will be able to:

1. Determine velocity and acceleration components using cylindrical coordinates.

In-Class Activities:

- Check Homework
- Reading Quiz
- Applications
- Velocity Components
- Acceleration Components
- Concept Quiz
- Group Problem Solving
- Attention Quiz

READING QUIZ

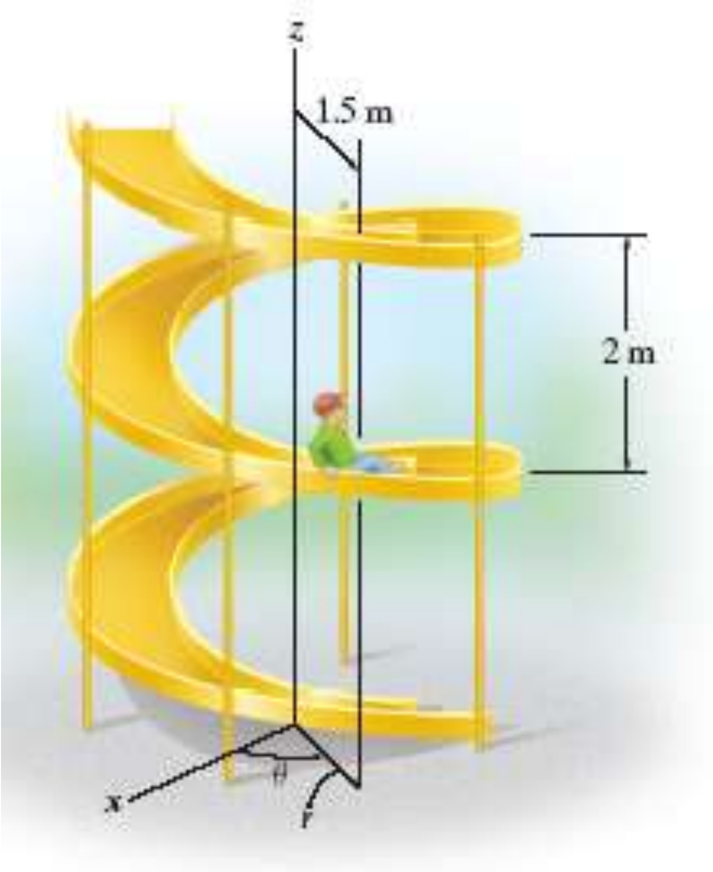
1. In a polar coordinate system, the velocity vector can be written as $\boldsymbol{v} = v_r\boldsymbol{u_r} + v_\theta\boldsymbol{u_\theta} = \dot{r}\boldsymbol{u_r} + r\dot{\theta}\boldsymbol{u_\theta}$. The term $\dot{\theta}$ is called

   A) transverse velocity.
   B) radial velocity.
   C) angular velocity.
   D) angular acceleration.

2. The speed of a particle in a cylindrical coordinate system is

   A) $\dot{r}$
   B) $r\dot{\theta}$
   C) $\sqrt{(r\dot{\theta})^2 + (\dot{r})^2}$
   D) $\sqrt{(r\dot{\theta})^2 + (\dot{r})^2 + (\dot{z})^2}$

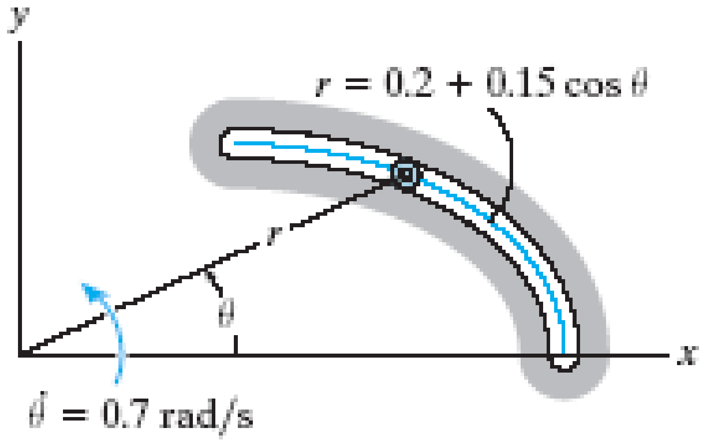
APPLICATIONS



The cylindrical coordinate system is used in cases where the particle moves along a 3-D curve.

In the figure shown, the boy slides down the slide at a constant speed of 2 m/s. How fast is his elevation from the ground changing (i.e., what is $\dot{z}$)?

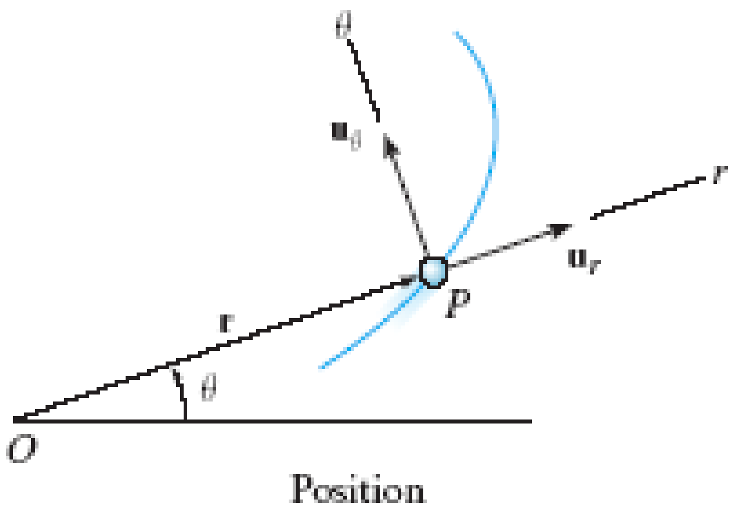
APPLICATIONS
(continued)



A polar coordinate system is a 2-D representation of the cylindrical coordinate system.

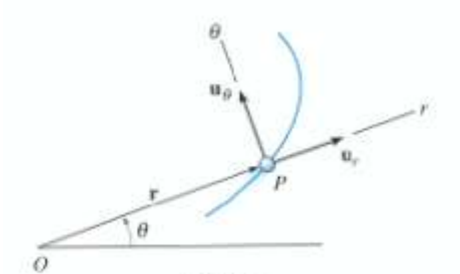
When the particle moves in a plane (2-D), and the radial distance, r, is not constant, the polar coordinate system can be used to express the path of motion of the particle.

CYLINDRICAL COMPONENTS
(Section 12.8)

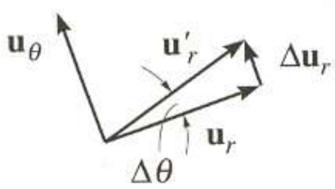


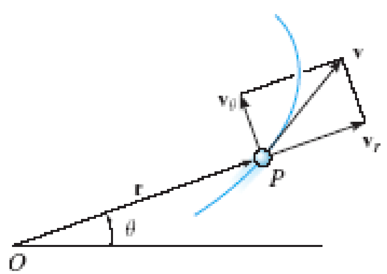
We can express the location of P in polar coordinates as $\boldsymbol{r} = \mathrm{r}\boldsymbol{u}_r$. Note that the radial direction, r, extends outward from the fixed origin, O, and the transverse coordinate, θ, is measured counter-clockwise (CCW) from the horizontal.

VELOCITY (POLAR COORDINATES)



Position





Velocity

The instantaneous velocity is defined as:

$$\boldsymbol{v} = d\boldsymbol{r}/dt = d(r\boldsymbol{u_r})/dt$$

$$\boldsymbol{v} = \dot{r}\boldsymbol{u_r} + r\,\frac{d\boldsymbol{u_r}}{dt}$$

Using the chain rule:

$$d\boldsymbol{u_r}/dt = (d\boldsymbol{u_r}/d\theta)(d\theta/dt)$$

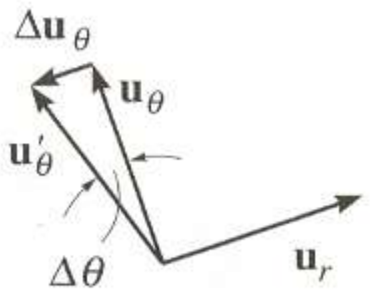
We can prove that $d\boldsymbol{u_r}/d\theta = \boldsymbol{u_\theta}$ so $d\boldsymbol{u_r}/dt = \dot{\theta}\boldsymbol{u_\theta}$

Therefore: $\boldsymbol{v} = \dot{r}\boldsymbol{u_r} + r\dot{\theta}\boldsymbol{u_\theta}$

Thus, the velocity vector has two components: $\dot{r}$, called the radial component, and $r\dot{\theta}$, called the transverse component. The speed of the particle at any given instant is the sum of the squares of both components or

$$v = \sqrt{(r\,\dot{\theta}\,)^2 + (\,\dot{r}\,)^2}$$

ACCELERATION (POLAR COORDINATES)



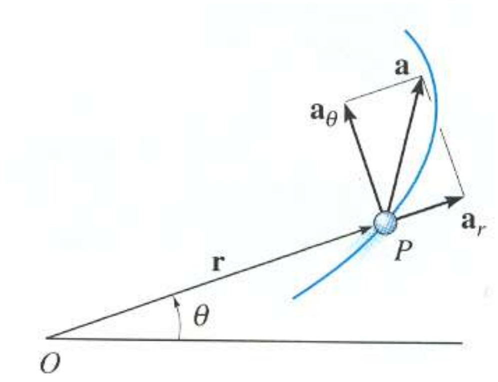
$$d\boldsymbol{u}_\theta/dt = (d\boldsymbol{u}_\theta/d\theta)(d\theta/dt)$$
$$= -\boldsymbol{u}_r\dot{\theta}$$

The instantaneous acceleration is defined as:

$$\boldsymbol{a} = d\boldsymbol{v}/dt = (d/dt)(\dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta)$$

After manipulation, the acceleration can be expressed as

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta$$



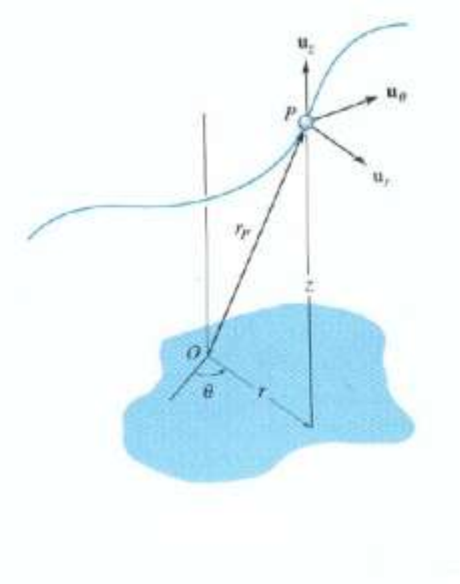
Acceleration

The term $(\ddot{r} - r\dot{\theta}^2)$ is the radial acceleration or a_r.

The term $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ is the transverse acceleration or a_θ

The magnitude of acceleration is $a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})^2}$

CYLINDRICAL COORDINATES



If the particle P moves along a space curve, its position can be written as

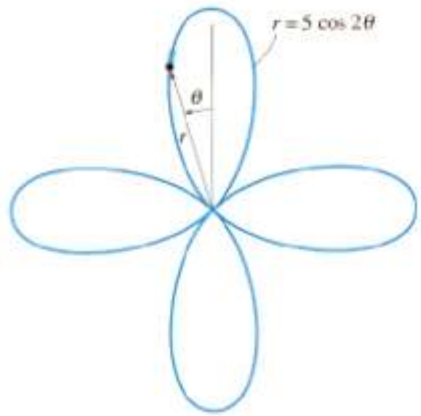
$$\boldsymbol{r}_P = r\boldsymbol{u}_r + z\boldsymbol{u}_z$$

Taking time derivatives and using the chain rule:

Velocity: $\boldsymbol{v}_P = \dot{r}\boldsymbol{u}_r + r\dot{\theta}\boldsymbol{u}_\theta + \dot{z}\boldsymbol{u}_z$

Acceleration: $\boldsymbol{a}_P = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u}_\theta + \ddot{z}\boldsymbol{u}_z$

EXAMPLE



Given: $r = 5\cos(2\theta)$ (m)

$\dot{\theta} = 3t^2$ (rad/s)

$\theta_o = 0$

Find: Velocity and acceleration at $\theta = 30°$.

Plan: Apply chain rule to determine $\dot{r}$ and $\ddot{r}$ and evaluate at $\theta = 30°$.

Solution:

$$\theta = \int_{t_o = 0}^{t} \dot{\theta}\, dt = \int_{0}^{t} 3t^2\, dt = t^3$$

At $\theta = 30°$, $\theta = \dfrac{\pi}{6} = t^3$. Therefore: $t = 0.806$ s.

$$\dot{\theta} = 3t^2 = 3(0.806)^2 = 1.95 \text{ rad/s}$$

EXAMPLE
(continued)

$$\ddot{\theta} = 6t = 6(0.806) = 4.836 \text{ rad/s}^2$$

$$r = 5\cos(2\theta) = 5\cos(60) = 2.5\text{m}$$

$$\dot{r} = -10\sin(2\theta)\dot{\theta} = -10\sin(60)(1.95) = -16.88 \text{ m/s}$$

$$\ddot{r} = -20\cos(2\theta)\dot{\theta}^2 - 10\sin(2\theta)\ddot{\theta}$$

$$= -20\cos(60)(1.95)^2 - 10\sin(60)(4.836) = -80 \text{ m/s}^2$$

Substitute in the equation for velocity

$$\boldsymbol{v} = \dot{r}\boldsymbol{u_r} + r\dot{\theta}\boldsymbol{u_\theta}$$

$$\boldsymbol{v} = -16.88\boldsymbol{u_r} + 2.5(1.95)\boldsymbol{u_\theta}$$

$$v = \sqrt{(16.88)^2 + (4.87)^2} = 17.57 \text{ m/s}$$

EXAMPLE
(continued)

Substitute in the equation for acceleration:

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u_\theta}$$

$$\boldsymbol{a} = [-80 - 2.5(1.95)^2]\boldsymbol{u_r} + [2.5(4.836) + 2(-16.88)(1.95)]\boldsymbol{u_\theta}$$

$$\boldsymbol{a} = -89.5\boldsymbol{u_r} - 53.7\boldsymbol{u_\theta} \text{ m/s}^2$$

$$a = \sqrt{(89.5)^2 + (53.7)^2} = 104.4 \text{ m/s}^2$$

CONCEPT QUIZ

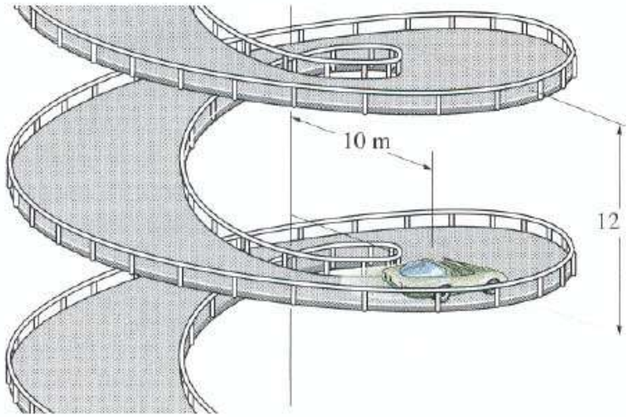
1. If $\dot{r}$ is zero for a particle, the particle is

   A) not moving.
   B) moving in a circular path.
   C) moving on a straight line.
   D) moving with constant velocity.

2. If a particle moves in a circular path with constant velocity, its radial acceleration is

   A) zero.
   B) $\ddot{r}$.
   C) $-r\dot{\theta}^2$.
   D) $2\dot{r}\dot{\theta}$.

GROUP PROBLEM SOLVING



Given: The car's speed is constant at 1.5 m/s.

Find: The car's acceleration (as a vector).

Hint: The tangent to the ramp at any point is at an angle

$$\phi = \tan^{-1}\left(\frac{12}{2\pi(10)}\right) = 10.81$$

Also, what is the relationship between ϕ and θ?

Plan: Use cylindrical coordinates. Since r is constant, all derivatives of r will be zero.

Solution: Since r is constant the velocity only has 2 components:

$$v_\theta = r\dot{\theta} = v\cos\phi \quad \text{and} \quad v_z = \dot{z} = v\sin\phi$$

GROUP PROBLEM SOLVING (continued)

Therefore: $\dot{\theta} = \left(\dfrac{v\cos\phi}{r}\right) = 0.147 \text{ rad/s}$

$$\ddot{\theta} = 0$$

$$v_z = \dot{z} = v\sin\phi = 0.281 \text{ m/s}$$

$$\ddot{z} = 0$$

$$\dot{r} = \ddot{r} = 0$$

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{u_\theta} + \ddot{z}\boldsymbol{u_z}$$

$$\boldsymbol{a} = (-r\dot{\theta}^2)\boldsymbol{u_r} = -10(0.147)^2\boldsymbol{u_r} = -0.217\boldsymbol{u_r} \text{ m/s}^2$$

ATTENTION QUIZ

1. The radial component of velocity of a particle moving in a circular path is always

   A) zero.

   B) constant.

   C) greater than its transverse component.

   D) less than its transverse component.

2. The radial component of acceleration of a particle moving in a circular path is always

   A) negative.

   B) directed toward the center of the path.

   C) perpendicular to the transverse component of acceleration.

   D) All of the above.