Centre of Mass

$$\vec{R}_{cm} = \frac{1}{M_{Total}} \sum m_i \vec{r}_i \qquad x_{cm} = \frac{1}{M_{Total}} \sum m_i x_i \qquad y_{cm} = \frac{1}{M_{Total}} \sum m_i y_i \qquad z_{cm} = \frac{1}{M_{Total}} \sum m_i z_i$$

The Centre of Mass is the point where all you could balance an object on one finger. It is as if all the mass were at that point. From play in the park we know how the CM works. For example if two children of the same mass were on a seesaw, the pivot would need to be under the centre of the seesaw for the two children to balance. This illustrated in figure 1(a) below. If one of the children is much heavier, the pivot must be closer to the heavier child for the two to balance as shown in figure 1(b) below. If we ignore the mass of the seesaw, we can easily find the where to place the pivot by meeting the requirement that $ML_1 = mL_2$ where $L_1 + L_2 = L$, where L is the distance separating the two children. This requirement yields $L_1 = \frac{m}{M+m}L$ and $L_2 = \frac{M}{M+m}L$. The equation for x_{cm} above will give the same result. Take the heavier child to be the origin 0 and the light child to be at L, the formula gives

$$x_{cm} = \frac{1}{M_{Total}} \sum m_i x_i = \frac{1}{M+m}(M \times 0 + m \times L) = \frac{m}{M+m}L$$

The same result as before.

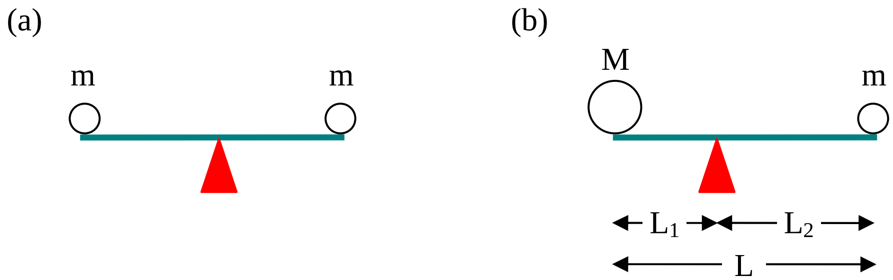

Figure 1. Two children on a seesaw in the park.

1. Find the CM for the following 1D shapes:

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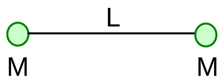
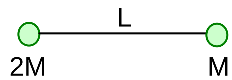
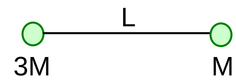
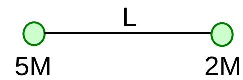
<tr>
<td>(a) </td>
<td>(b) </td>
<td>(c) </td>
<td>(d) </td>
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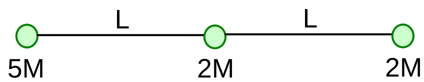
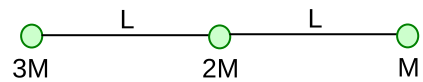
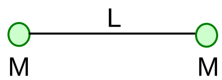
<td colspan="2">(e) </td>
<td colspan="2">(f) </td>
</tr>
</table>

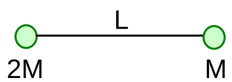
SOLUTIONS

1(a).



Symmetry says the CM is halfway between the identical masses.

1(b).

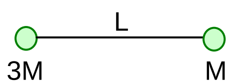


The total mass is 3M; the fractions are 2/3 and 1/3. The CM must be closer to the 2M mass. CM is ($^1/_3$)L from lefthand mass.

Or using the equation with the 2M mass as the origin:

$$x_{cm} = \frac{1}{2\mathrm{M}+\mathrm{M}}\left(2\mathrm{M}\times 0+\mathrm{M}\times\mathrm{L}\right)=\frac{1}{3}\mathrm{L}$$

1(c).

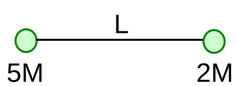


The total mass is 4M; the fractions are 3/4 and 1/4. The CM must be closer to the 3M mass. CM is ($^1/_4$)L from lefthand mass.

Or using the equation with the 3M mass as the origin:

$$x_{cm} = \frac{1}{3\mathrm{M}+\mathrm{M}}\left(3\mathrm{M}\times 0+\mathrm{M}\times\mathrm{L}\right)=\frac{1}{4}\mathrm{L}$$

1(d).

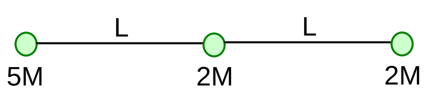


The total mass is 7M; the fractions are 5/7 and 2/7. The CM must be closer to the 5M mass. CM is ($^2/_7$)L from lefthand mass.

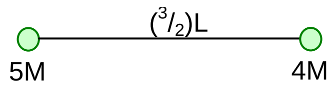
Or using the equation with the 5M mass as the origin:

$$x_{cm} = \frac{1}{5\mathrm{M}+2\mathrm{M}}\left(5\mathrm{M}\times 0+2\mathrm{M}\times\mathrm{L}\right)=\frac{2}{7}\mathrm{L}$$

1(e).



With more than two masses we can work with two at a time. Clearly for the two 2M masses, their CM is directly between them. So the diagram for 1(e) reduces to:

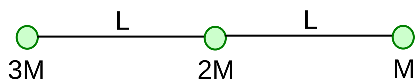


Note that the total mass of the two 2M masses (i.e. 4M) acts as if it were $({}^{3}/_{2})$L from the 5M mass. Finally the total CM is $({}^{4}/_{9})({}^{3}/_{2})\mathrm{L} = ({}^{2}/_{3})$L from the 5M mass.

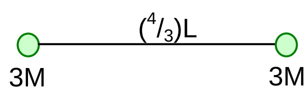
Or using the equation with the 5M mass as the origin:

$$x_{cm} = \frac{1}{5\mathrm{M}+2\mathrm{M}+2\mathrm{M}}\left(5\mathrm{M}\times 0+2\mathrm{M}\times \mathrm{L}+2\mathrm{M}\times 2\mathrm{L}\right)=\frac{6}{9}\mathrm{L}=\frac{2}{3}\mathrm{L}$$

1(f).



With more than two masses we can work with two at a time. Clearly for the 2M and M masses, their CM is $({}^{1}/_{3})$L from the 2M mass. So the diagram for 1(f) reduces to:

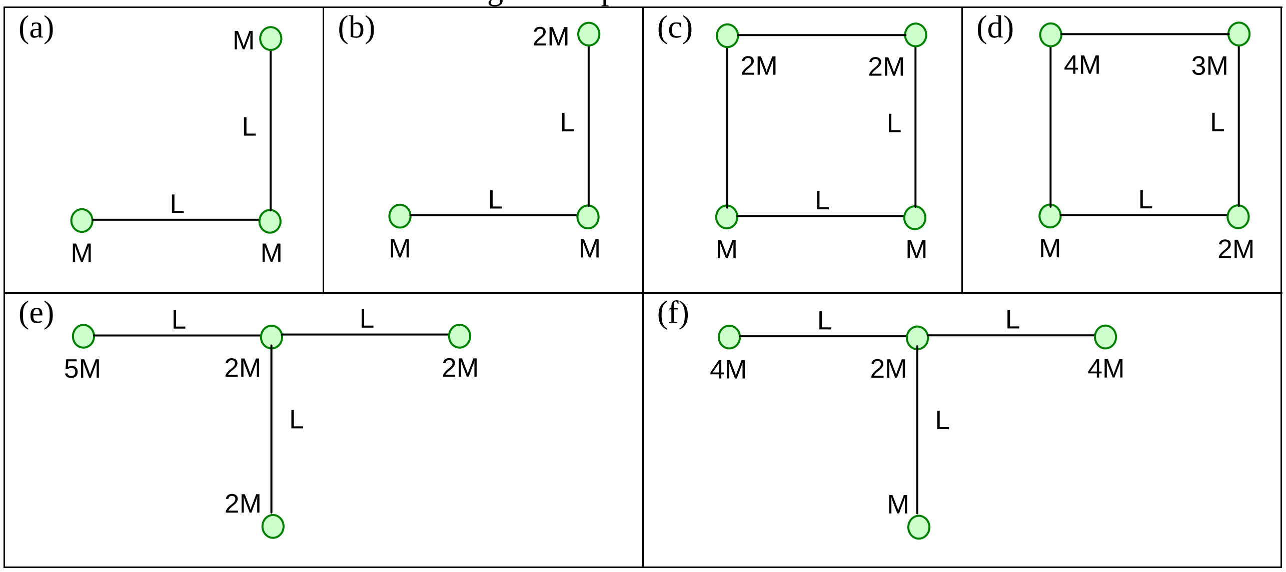


Note that the total mass of the 2M and M masses (i.e. 3M) acts as if it were $({}^{4}/_{3})$L from the 3M mass. Finally the total CM is exactly in the middle $({}^{1}/_{2})({}^{4}/_{3})\mathrm{L} = ({}^{2}/_{3})$L from the leftmost 3M mass.

Or using the equation with the 3M mass as the origin:

$$x_{cm} = \frac{1}{3\mathrm{M}+2\mathrm{M}+\mathrm{M}}\left(3\mathrm{M}\times 0+2\mathrm{M}\times \mathrm{L}+\mathrm{M}\times 2\mathrm{L}\right)=\frac{4}{6}\mathrm{L}=\frac{2}{3}\mathrm{L}$$

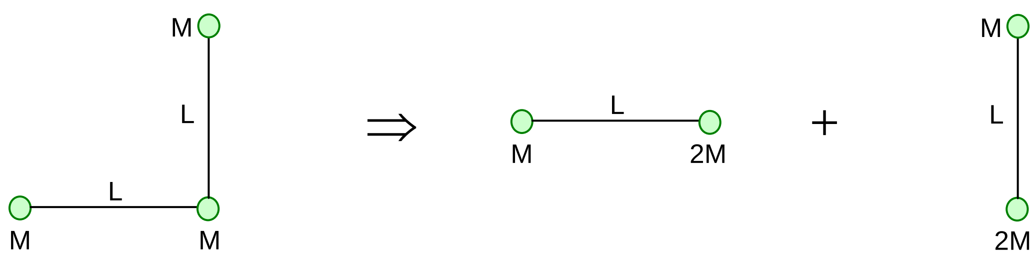
2. Find the CM for the following 2D shapes:



SOLUTIONS

These are 2D problems and we have to treat the x and y CM's independently. That is we find out how much mass is at each value of x and how much mass is at each value of y.

2(a).



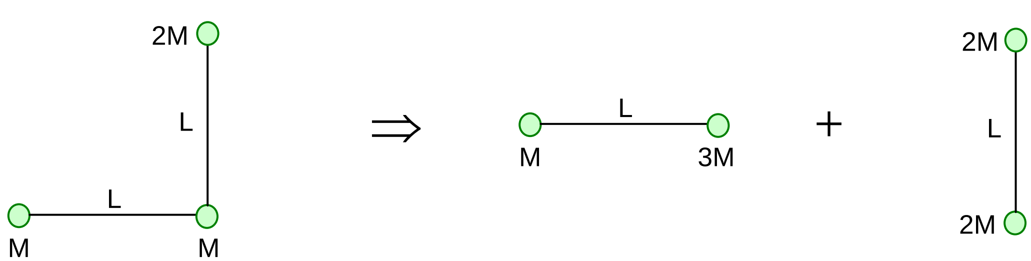
Note that there are two masses of M with the same value of x = L, and two masses of M with the same value of y = 0. The CM is thus at $\{(^2/_3)\text{L}, (^1/_3)\text{L}\}$ from the lower left mass M.

Or using the equations with the lower left mass M as the origin:

$$x_{cm} = \frac{1}{\text{M}+\text{M}+\text{M}}\left(\text{M}\times 0+\text{M}\times\text{L}+\text{M}\times\text{L}\right)=\frac{2}{3}\text{L}\,,$$

$$y_{cm} = \frac{1}{\text{M}+\text{M}+\text{M}}\left(\text{M}\times 0+\text{M}\times 0+\text{M}\times\text{L}\right)=\frac{1}{3}\text{L}\,.$$

2(b).



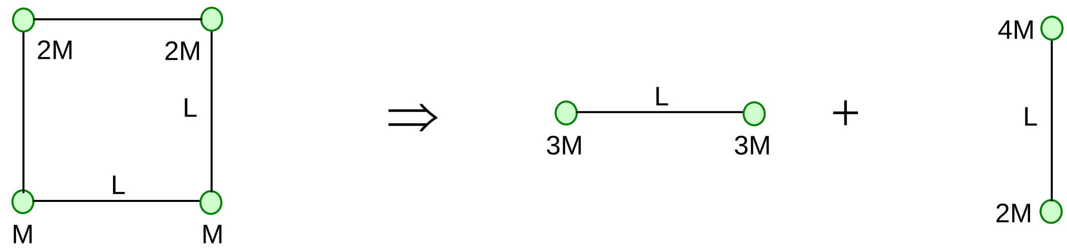
Note that there are two masses, M and 2M, with the same value of x = L, and two masses of M with the same value of y = 0. The CM is thus at {($^3/_4$)L, ($^1/_2$)L} from the lower left mass M.

Or using the equations with the lower left mass M as the origin:

$$x_{cm} = \frac{1}{\mathrm{M}+\mathrm{M}+2\mathrm{M}}\left(\mathrm{M}\times 0+\mathrm{M}\times \mathrm{L}+2\mathrm{M}\times \mathrm{L}\right)=\frac{3}{4}\mathrm{L}\,,$$

$$y_{cm} = \frac{1}{\mathrm{M}+\mathrm{M}+2\mathrm{M}}\left(\mathrm{M}\times 0+\mathrm{M}\times 0+2\mathrm{M}\times \mathrm{L}\right)=\frac{1}{2}\mathrm{L}\,.$$

2(c).



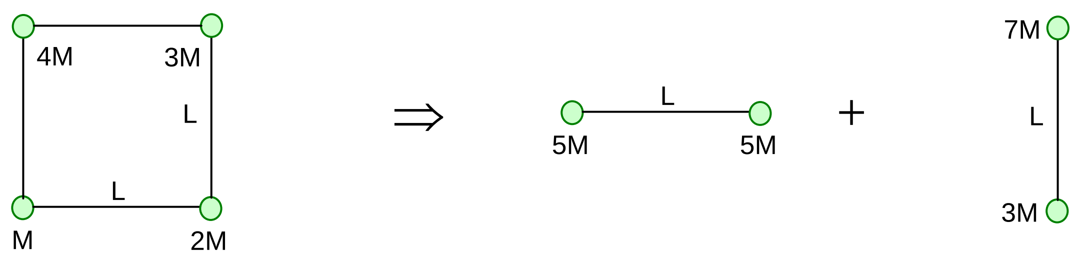
Note that there are two masses, M and 2M, with the same value of x = 0 and two masses, M and 2M, with the same value of x = L. As well there are two masses of M with the same value of y = 0 and two masses of 2M with the same value of y = L. The CM is thus at {($^1/_2$)L, ($^2/_3$)L} from the lower left mass M.

Or using the equations with the lower left mass M as the origin:

$$x_{cm} = \frac{1}{\mathrm{M}+\mathrm{M}+2\mathrm{M}+2\mathrm{M}}\left(\mathrm{M}\times 0+2\mathrm{M}\times 0+\mathrm{M}\times \mathrm{L}+2\mathrm{M}\times \mathrm{L}\right)=\frac{1}{2}\mathrm{L}\,,$$

$$y_{cm} = \frac{1}{\mathrm{M}+\mathrm{M}+2\mathrm{M}+2\mathrm{M}}\left(\mathrm{M}\times 0+2\mathrm{M}\times \mathrm{L}+\mathrm{M}\times 0+2\mathrm{M}\times \mathrm{L}\right)=\frac{2}{3}\mathrm{L}\,.$$

2(d).

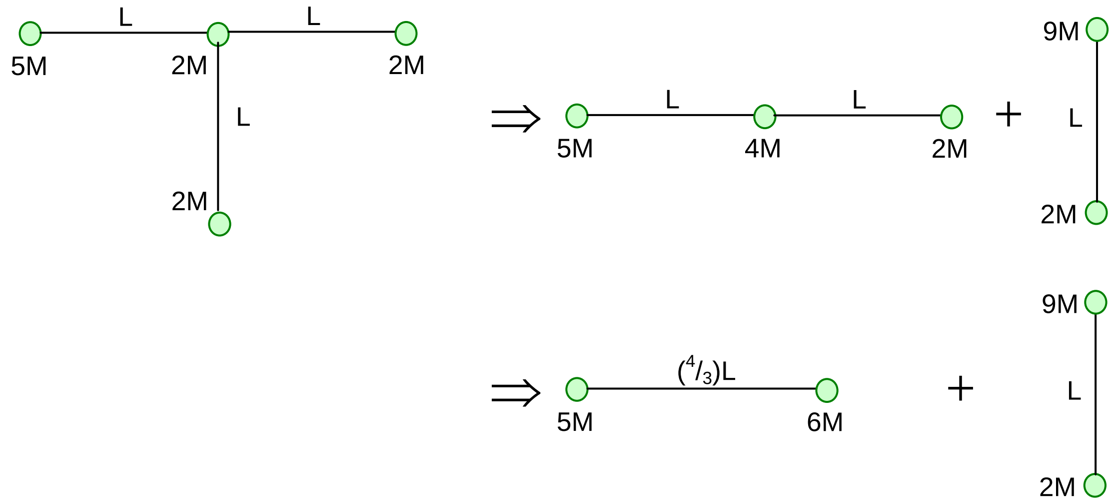


Note that there are two masses, M and 4M, with the same value of x = 0 and two masses, 2M and 3M, with the same value of x = L. As well there are two masses, M and 2M, with the same value of y = 0 and two masse, 4M and 3M, with the same value of y = L. The CM is thus at {($^1/_2$)L, ($^7/_{10}$)L} from the lower left mass M.

Or using the equations with the lower left mass M as the origin:

$$x_{cm} = \frac{1}{M+2M+3M+4M}(M\times 0 + 4M\times 0 + 2M\times L + 3M\times L) = \frac{1}{2}L\,,$$
$$y_{cm} = \frac{1}{M+2M+3M+4M}(M\times 0 + 4M\times L + 2M\times 0 + 3M\times L) = \frac{7}{10}L\,.$$

2(e).

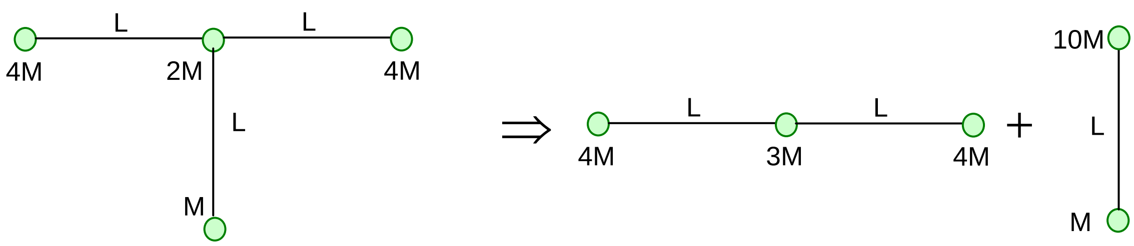


Note that there are three different x positions involved. There are two masses, 2M and 2M equivalent to 4M, with the same value of x = L and one mass, 2M, at 2L as shown above. The CM for these two is $(^1/_3)$L from the 4M mass. Now we have the equivalent of a 5M and a 6M mass separated by $(^4/_3)$L. Thus, in the x-direction, for the total object, the CM is at $(^6/_{11})(^4/_3)$L = $(^8/_{11})$L from the 5M mass. Vertically the CM is at $(^2/_{11})$L from the upper masses. The CM is thus at $\{(^8/_{11})L, -(^2/_{11})L\}$ from the lower left mass 5M.

Or using the equations with the lower left mass 5M as the origin:

$$x_{cm} = \frac{1}{5M+2M+2M+2M}(5M\times 0 + 2M\times L + 2M\times L + 2M\times 2L) = \frac{8}{11}L\,,$$
$$y_{cm} = \frac{1}{5M+2M+2M+2M}(5M\times 0 + 2M\times 0 + 2M\times(-L) + 2M\times 0) = -\frac{2}{10}L\,.$$

2(f).



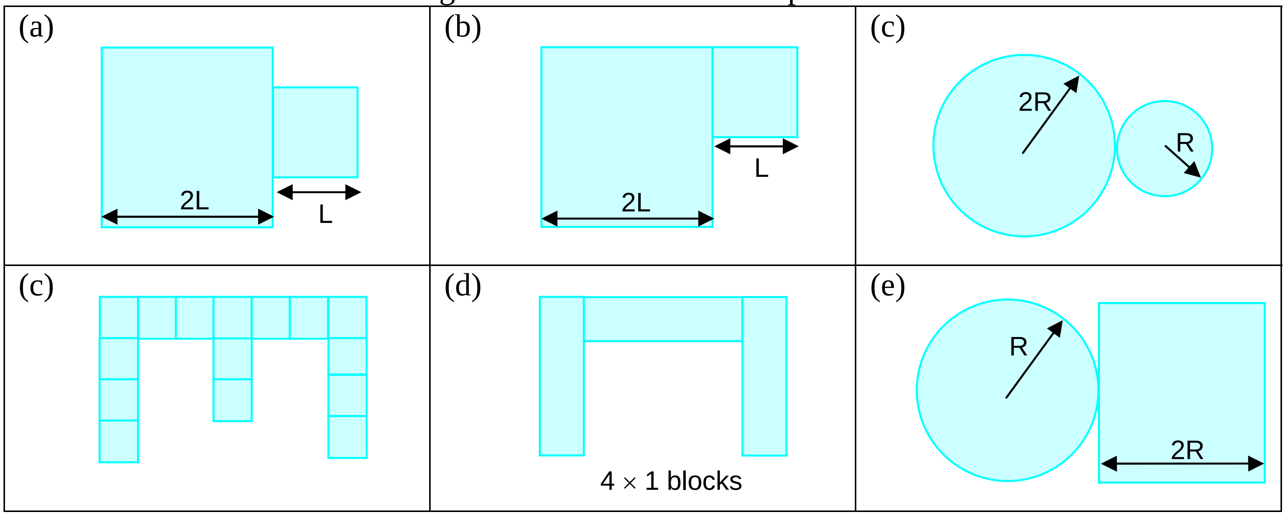
The object is symmetric in the x-direction the CM is located on the axis containing the masses 2M and M. Vertically the CM is at $(^1/_{11})$L from the upper masses. The CM is thus at $\{L, -(^1/_{11})L\}$ from the upper left mass 4M.

Or using the equations with the upper left mass 4M as the origin:

$$x_{cm} = \frac{1}{4\mathrm{M}+2\mathrm{M}+\mathrm{M}+4\mathrm{M}}\left(4\mathrm{M}\times 0+2\mathrm{M}\times \mathrm{L}+\mathrm{M}\times \mathrm{L}+4\mathrm{M}\times 2\mathrm{L}\right)=\mathrm{L}\ ,$$

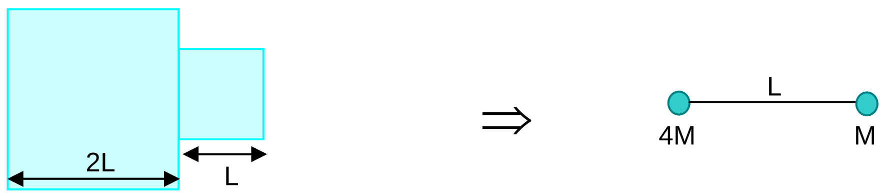
$$y_{cm} = \frac{1}{4\mathrm{M}+2\mathrm{M}+\mathrm{M}+4\mathrm{M}}\left(\mathrm{M}\times 0+2\mathrm{M}\times 0+\mathrm{M}\times (-\mathrm{L})+4\mathrm{M}\times 0\right)=-\frac{1}{11}\mathrm{L}\ .$$

3. Find the CM for the following extended uniform 2D shapes:



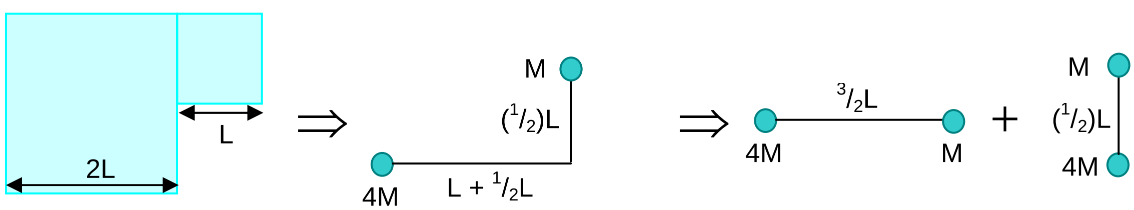
For uniform signs mass is proportional to area. Note for a rectangle Area = Length × Height, for a circle Area = πR^2. We first reduce simple shapes to point masses.

3(a).



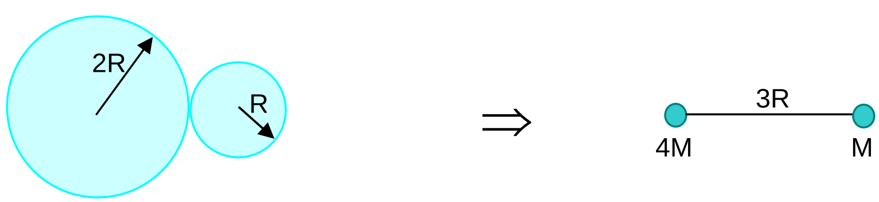
Note that the large square has four times the area, and thus four times the mass, of the little square. The CM is $(^1/_5)L$ from the centre of the larger square or $\{(^6/_5)L, L\}$ from the lower left corner of the shape.

3(b).



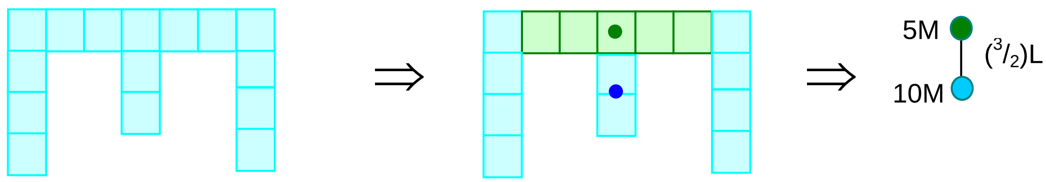
Note that the large square has four times the area, and thus four times the mass, of the little square. The CM is $(^1/_5)(^3/_2)L = (^3/_{10})L$ from the centre of the larger square and vertically up $(^1/_5)(^1/_2)L$. From the lower left corner of the shape the CM is at $\{(^{13}/_{10})L, (^{11}/_{10})L\}$.

3(c).



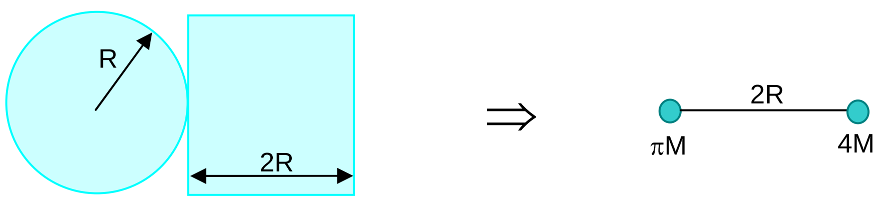
Note that the large circle has four times the area, and thus four times the mass, of the little circle. The centres of the circles are 3R apart. Thus the CM is $({}^1/_5)(3R) = ({}^3/_5)R$ from the centre of the larger circle or $({}^8/_5)R$ from the left edge of the large circle.

3(d).



Each small square has the same mass and side size L. Notice that the shape is symmetric about a vertical axis through the centre. Next I chose to break the original object into two simpler shapes, the blue and the green, which exploit the symmetry of the problem. The dots show the respective CMs. The blue shape has 10 squares and the green rectangle has 5 squares, so the masses are 10M and 5M respectively. The CMs of the two pieces are $({}^3/_2)L$ apart. The CM is $({}^5/_{15})\,({}^3/_2)L = ({}^1/_2)L$ above the centre of the blue shape (the blue dot). From the lower left hand corner this is $\{(3{}^1/_2)L, (2{}^1/_2)L\}$.

3(e).



The area of the circle is πR^2 and the area of the square is $4R^2$. The ratio of the masses of each shape is thus π:4. The CM is closer to the centre of the square since 4 is greater than π. The CM is located $\frac{4}{4+\pi}2R$ from the centre of the circle or $R+\frac{8}{4+\pi}R$ from the left edge of the circle.