Uncertainty Propagation

Addition and Subtraction

For any combination of addition or subtraction, the absolute error in the result is equal to the sum of the absolute uncertainties of the parts.

If $F = A - B + C + D$, then $\delta F = \delta A + \delta B + \delta C + \delta D$.

Powers and Roots

When a measurement *A* is raised to a power *z*, the relative uncertainty in the result is *z* time the relative uncertainty in *A*.

If $F = A^z$, $\frac{\delta F}{F} = z\frac{\delta A}{A}$.

Multiplication and Division

For any combination of multiplication or division, the relative error in the result is equal to the sum of the absolute uncertainties of the parts.

If $F = \frac{AB}{CD}$, then $\frac{\delta F}{F} = \frac{\delta A}{A} + \frac{\delta B}{B} + \frac{\delta C}{C} + \frac{\delta D}{D}$.

Functions

The absolute uncertainty in a function of a measurement *A* is equal to the absolute uncertainty of the measurement multiplied by the derivative of the function.

If $F = f(A)$, then $\delta F = \delta A \times \left.\frac{df(x)}{dx}\right|_{x=A}$.

Table 1 Common functions and their derivatives. Note: Uncertainty in angles must be in radians.

f(x)	*f′(x)*	*f(x)*	*f′(x)*
sin(x)	*cos(x)*	*arcsin(x)*	$\frac{1}{\sqrt{1-x^2}}$
cos(x)	*-sin(x)*	*arccos(x)*	$-\frac{1}{\sqrt{1-x^2}}$
tan(x)	$\frac{1}{\cos^2(x)}$	*arctan(x)*	$\frac{1}{1+x^2}$
ln(x)	$\frac{1}{x}$	e^x	e^x

Examples involving the use of a single rule

1. Find $Z = W - X - Y$, where $W = 0.00123 \pm 0.00007$, $X = 0.0032 \pm 0.0008$, and $Y = -0.0061 \pm 0.0009$.

   The principle, or measured part, of Z is

   $$Z = 0.00123 - 0.0032 - -0.0061 = 0.00413 .$$

   Since finding Z only involves using one operation, addition or subtraction, we know the uncertainty in Z is

   $$\delta Z = 0.00007 + 0.0008 + 0.0009 = 0.00177 .$$

   Thus our result, to the correct number of significant figures is, $Z = 0.0041 \pm 0.0018$.

2. Find $\beta = arcsin(x)$, where $x = 0.642 \pm 0.017$.

   The principle, or measured part, of β is

   $$\beta = arcsin(0.642)= 0.69710 \text{ rad or } 39.941^{\circ}.$$

   Since finding β only involves using one function, the uncertainty in β is

   $$\delta\beta = 0.017 \times 1/(1 - 0.017^{2})^{½} = 0.07700 \text{ rad or } 0.974^{\circ}.$$

   Thus our result, to the correct number of significant figures is, $\beta = 40.0 \pm 1.0^{\circ}$.

3. Find $U = mgh$ where $m = 0.502 \pm 0.005$, $g =0.981 \pm 0.001$, and $h = -2.15 \pm 0.05$.

   The principle, or measured part, of U is

   $$U = 0.502 \times 0.981 \times 2.15 = 10.588 .$$

   Since finding U only involves using one operation, multiplication, we know that the relative uncertainty in U is

   $$\delta U/U = 0.005/0.502 + 0.001/9.81 + 0.05/|-2.15| = 0.0333 \text{ or } 3.33\% .$$

   Note that absolute and relative uncertainties are always positive quantities. To find the absolute uncertainty, we multiply the U by its relative uncertainty, $\delta U/U$. So

   $$\delta U = 0.0333 \times 10.588 = 0.35277.$$

   Thus our result, to the correct number of significant figures is, $U = 10.6 \pm 0.4$.

4. Find $Y = X^{-½}$ where $X = 3.25 \pm 0.08$.

   The principle, or measured part, of Y is

$$Y = (3.25)^{-½} = 0.544700.$$

   Since finding Y only involves using one operation, multiplication, we know that the relative uncertainty in Y is

$$\delta Y/Y = |-½\ 0.08/3.25| = 0.00123 \text{ or } 0.123\% .$$

   Note that absolute and relative uncertainties are always positive quantities. To find the absolute uncertainty, we multiply the Y by its relative uncertainty, $\delta Y/Y$. So

$$\delta Y = 0.00123 \times 0.544700 = 0.00067.$$

   Thus our result, to the correct number of significant figures is, $Y = 0.5447 \pm 0.0007$.

Examples involving exact constants

If a formula involves the addition or subtraction of an exact number, the exact number does not contribute to the result since it has no uncertainty. This makes sense. Consider $B = X + 2$, then $\delta B = \delta X + \delta[2] = \delta X + 0 = \delta X$.

5. Find the uncertainty in $Y = e^X + \pi$, where $X = 1.23 \pm 0.03$. We can ignore the exact number π. Thus $\delta Y = \delta X\, e^X = 0.03 \times e^{1.23} = 0.103$.

If a formula involves an exact number as a multiplicative factor, the uncertainty also gets multiplied by the same exact factor. That is, if $B = 2X$, then $\delta B = 2\delta X$. This makes sense, since if we rewrite B as $B = X + X$, it would be clear that $\delta B = \delta X + \delta X = 2\delta X$.

6. Find the uncertainty in $Y = \pi e^X$, where $X = 1.23 \pm 0.03$. We need to multiply by the exact factor π. Thus $\delta Y = \pi \times \delta X\, e^X = \pi \times 0.03 \times e^{1.23} = 0.322$.

Combining Uncertainty Rules

More complicated equations are easy to do. We only need think about the order in which we do the mathematical operations involved. For example, consider the equation $F = A^{½} - B\cos(\theta)$, where $A = 12.65$, $B = 4.88$, and $\theta = 13.7^o$. Most people would do the following

a. Calculate the value of $A^{½}$.
b. Calculate the value of $\cos(\theta)$.
c. Calculate the value of $B \times \cos(\theta)$.
d. Subtract the value of $B\cos(\theta)$ from the value of $A^{½}$.

In dealing with measured quantities, with we do the same but also do a one-rule uncertainty calculation. That is, if $A = 12.65 \pm 0.07$, $B = 4.88 \pm 0.05$, and $\theta = 13.7 \pm 0.3^\circ$, we would do the following

a. Calculate the value of $A^{½}$.
   Calculate the uncertainty in $A^{½}$.
b. Calculate the value of $cos(\theta)$.
   Calculate the uncertainty in $cos(\theta)$.
c. Calculate the value of $B \times cos(\theta)$.
   Calculate the uncertainty in $Bcos(\theta)$.
d. Subtract the value of $Bcos(\theta)$ from the value of $A^{½}$.
   Calculate the uncertainty in $A^{½} - Bcos(\theta)$.

Doing these steps for our example $F = A^{½} - Bcos(\theta)$.

First the principle value of F is

$$F = (12.65)^{½} - (4.88)cos(13.7^\circ) = -1.184.$$

a. To get the uncertainty, first let $X = A^{½}$. We know

$$\delta X/X = ½\delta A/A = ½(0.07/12.65) = 0.00277 = 0.277\%.$$

   As a result, the absolute uncertainty is $\delta X = 0.00277 \times (12.65)^{½} = 0.00985$.

b. So now $F = X - Bcos(\theta)$. Let $Y = cos(\theta)$, thus

$$\delta Y = \delta\theta\, sin(\theta) = 0.3 \times (\pi/180) \times sin(13.7^\circ) = 0.00124.$$

c. Now $F = X - BY$. Let $Z = BY$, thus

$$\delta Z/Z = \delta B/B + \delta Y/Y = (0.05/4.88) + (0.00124/0.97155) = 0.01152 = 1.152\%.$$

   The absolute uncertainty is $\delta Z = 0.01152 \times 4.88 \times cos(13.7^\circ) = 0.0546$.

e. Finally, $F = X - Z$. We know we are at the last step since this last operation will give us δF which is

$$\delta F = \delta X + \delta Z = 0.00985 + 0.0546 = 0.0645.$$

Thus $F = -1.18 \pm 0.06$ to the correct number of significant figures.

Spreadsheets

Spreadsheets can greatly assist with the numeric calculations, particularly if there is more than one set of data. First the data should go in one table. For each measurement, the principal value should go in one cell and the uncertainty in the next. Note that EXCEL works only with angles in radians, so angles and uncertainties in degrees should be converted. Second, all the uncertainty calculations should be done in a second table. Each step of the calculation above should go in a separate cell. See the example below. Note that the formulas are shown, not numerical values.

	A	B	C	D	E	F	G	H	I	J	K
1	Table1: The experimental data										
2			Uncertainty		Uncertainty		Uncertainty		Uncertainty		
3	Run	A	δA	B	δB	θ	δθ	θ	δθ		
4						(º)	(º)	(rad)	(rad)		
5	1	12.65	0.07	4.88	0.05	13.7	0.3	=F4*PI()/180	=G4*PI()/180		
6	2										
7	3										
8											
9	Table 2: Calculation of the uncertainty										
10											
11	Run	X=A^{½}	δX/X = ½δA/A	δX	Y = cos(θ)	δY = δθ sin(θ)	Z = BY	δZ/Z = δB/B + δY/Y	δZ	F	δF
12	1	=sqrt(B5)	=0.5*C5/B5	=C12*B12	=cos(H5)	=I5*sin(H5)	=D5*F12	=E5/D5+ F12/E12	=H12*G12	=B12-G12	=D12+I12
13	2										
14	3										
15											
16											