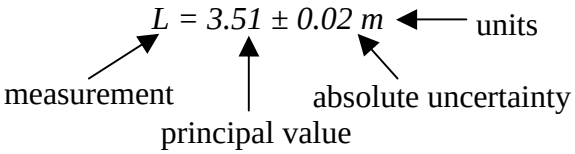
Measurement and Propagation of Uncertainty

Format and Significant Figures

Unlike exact numbers, measurements are ranges of values. Measurements are written to emphasize the centre of the possible range, also known as the principle value, and the width of the range about the principle value, known as the absolute uncertainty.



The absolute uncertainty should have only one significant figure. However, if the absolute uncertainty's significant figure is a 1, write two significant figures, e.g. $L = 3.513 \pm 0.016$.

The position of the last significant figure in the absolute uncertainty determines the least significant figure in the principal value. Any numbers to the right of the least significant figure are dropped when rounding to the correct number of significant digits.

Two measurements are said to agree within uncertainty if their ranges of possible values overlap.

Many instruments give the uncertainty as a relative or percentage uncertainty. A voltmeter may give its precision as 1%, so a reading of V = 9.75 Volts becomes a measurement

$$V = 9.75 \pm 1\% \times 9.75 = 9.75 \pm 0.0975 = 9.75 \pm 0.010 \text{ Volts}.$$

Absolute and relative uncertainties are always positive quantities.

Uncertainty Propagation Rules

In physics, one physical or measurable quantity may related to several others such as $F = ma$. Each of F, m, and a could be measured separately. But to compare F to ma we need to know how to find the uncertainty in the product of the two measurements m and a. For this reason we need to be able to use a number of rules for uncertainty propagation.

Addition and Subtraction

For any combination of addition or subtraction, the absolute error in the result is equal to the sum of the absolute uncertainties of the parts.

If $F = A - B + C + D$, then $\delta F = \delta A + \delta B + \delta C + \delta D$.

Multiplication and Division

For any combination of multiplication or division, the relative error in the result is equal to the sum of the absolute uncertainties of the parts.

If $F = \frac{AB}{CD}$, then $\frac{\delta F}{F} = \frac{\delta A}{A} + \frac{\delta B}{B} + \frac{\delta C}{C} + \frac{\delta D}{D}$.

Powers and Roots

When a measurement *A* is raised to a power *z*, the relative uncertainty in the result is *z* time the relative uncertainty in *A*.

If $F = A^z$, $\frac{\delta F}{F} = z\frac{\delta A}{A}$.

Functions

The absolute uncertainty in a function of a measurement *A* is equal to the absolute uncertainty of the measurement multiplied by the derivative of the function.

Given the function *f(x)*, then if $F = f(A)$, then $\delta F = \delta A \times f'(A)$, where $f'(x) = \frac{df}{dx}$.

Table 1 Common functions and their derivatives.

f(x)	*f′(x)*	*f(x)*	*f′(x)*
sin(x)	*cos(x)*	*arcsin(x)*	$\frac{1}{\sqrt{1-x^2}}$
cos(x)	*-sin(x)*	*arccos(x)*	$-\frac{1}{\sqrt{1-x^2}}$
tan(x)	$\frac{1}{\cos^2(x)}$	*arctan(x)*	$\frac{1}{1+x^2}$
ln(x)	$\frac{1}{x}$	e^x	e^x

Note. If *x* is an angle, δx must be in radians.

Combinations

Always do one operation at a time, work from the inside out following the *Order of Operations*. If $F = ln(A \times (B–C))$, first find $X = B – C$. Next find $Y = AX$. Finally evaluate $F = ln(Y)$.

Do not round until the end of the calculation!

Exercises

Round to the correct number of significant figures:

a) 17.4578 ± 0.0403 b) 8.2673 ± 0.0136 c) $(3.7689 \pm 0.0267) \times 10^{-4}$ d) 0.01567 ± 0.00074

Convert to absolute uncertainty:

a) $14.35 \pm 2\%$ b) $-0.895 \pm 1\%$ c) $2.7689 \times 10^{-4} \pm 5\%$ d) $0.006175 \pm 0.1\%$ e) $-1.450 \pm 1.1\%$

Which of the following agree within uncertainty:

a) $6.78 \pm 3\%$ *and* $7.02 \pm 1.5\%$
b) $(1.839 \pm 0.005) \times 10^{-4}$ *and* $(1.846 \pm 0.004) \times 10^{-4}$
c) -5.612 ± 0.007 *and* -5.625 ± 0.006
d) $1.111 \pm 0.2\%$ *and* $1.130 \pm 0.3\%$

Use a spreadsheet and uncertainty propagation rules to evaluate the following:

a) $F = \ln(A \times (B - C))$, where
   a) $A = 6.32 \pm 0.07$, $B = 4.55 \pm 0.05$, and $C = 2.71 \pm 0.03$
   b) $A = 3.13 \pm 0.03$, $B = 14.5 \pm 0.2$, and $C = 7.71 \pm 0.09$
   c) $A = 2.32 \pm 0.01$, $B = 9.23 \pm 0.11$, and $C = 4.17 \pm 0.08$

b) $F = X^2 \sin(\theta)$, where
   i. $X = 0.926 \pm 0.005$, $\theta = 22.4 \pm 0.6^\circ$
   ii. $X = 2.12 \pm 0.02$, $\theta = 37.3 \pm 0.2^\circ$
   iii. $X = 4.21 \pm 0.04$, $\theta = 15.1 \pm 0.3^\circ$

c) $F = A - BC^{1/3}$, where
   i. $A = 12.3 \pm 0.7$, $B = 4.55 \pm 0.05$, and $C = 162 \pm 5$
   ii. $A = 1.33 \pm 0.03$, $B = 0.145 \pm 0.002$, and $C = 7.71 \pm 0.09$
   iii. $A = 2.32 \pm 0.01$, $B = 9.23 \pm 0.11$, and $C = 4.17 \pm 0.08$